PEARSON

Pre-calculus 12

my WORKTEXT

Coming 2012

8 Permutations and Combinations

BUILDING ON

- listing outcomes of probability experiments
- solving equations

BIG IDEAS

- Counting strategies can be used to determine the number of ways to choose objects from a set or to arrange a set of objects.
- A permutation is an arrangement of a set of objects where order matters. A combination is a selection from a set of objects where order does not matter.
- Combinations can be used to expand a power of a binomial and to generate the terms in Pascal's triangle.

LEADING TO

- applying the properties of permutations and combinations to solve problems in probability

NEW VOCABULARY

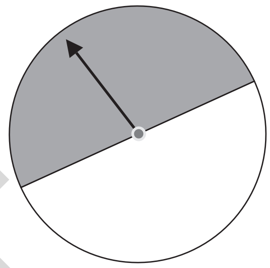
fundamental counting principle	combination
permutation	Pascal's triangle
factorial notation	binomial theorem

8.1 The Fundamental Counting Principle

FOCUS **Derive and apply the fundamental counting principle to solve problems.**

Get Started

A coin is tossed and the pointer on this spinner is spun.

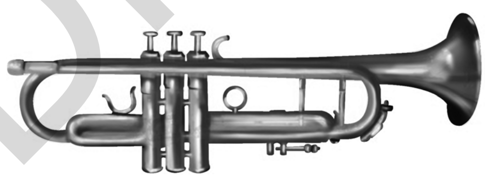


- List all the possible outcomes.
- What is the probability of each outcome?

Construct Understanding

How many ways can any or all of the 3 valves on a trumpet be completely pushed down?

List all possibilities systematically.



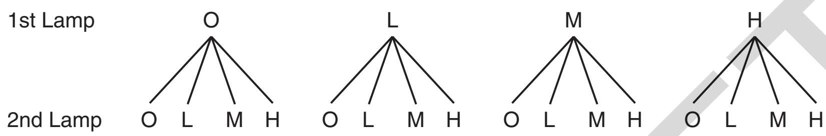
When it is necessary to list and count the number of possible choices or arrangements, graphic organizers can be useful.

Example 1 Using a Graphic Organizer to Determine the Number of Choices

A room has two lamps. Each lamp has 4 settings: off, low, medium, high. How many ways are there to set both lamps?

SOLUTION

Use a tree diagram.
Use letters to represent the possible settings.
O = Off L = Low M = Medium H = High



The tree diagram has 16 branches.
So, there are 16 ways to set both lamps.

Check Your Understanding

1. A fan has 3 settings: off, low, high. How many ways are there to set 3 fans?

THINK FURTHER

In *Example 1*, assume all lamp settings are equally likely. What is the probability that both lamps are on a high setting?

Example 1 illustrates the **fundamental counting principle**.
There are 4 ways to set the first lamp. For each of these ways, there are 4 ways to set the second lamp. So, the total number of ways is the product: $4 \cdot 4 = 16$
So, there are 16 ways to set both lamps.

The Fundamental Counting Principle

If there are n_1 different objects in one set and n_2 different objects in a second set, then the number of ways of choosing one object from each set is $n_1 \cdot n_2$.

This can be extended for k sets:
If there are n_1 different objects in one set, n_2 different objects in a second set, and so on, for k sets, then the number of ways of choosing one object from each set is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.

Example 2 Using the Counting Principle to Determine the Number of Choices

Check Your Understanding

2. For an online banking account, the minimum security standards require a password to have 2 letters followed by 5 digits. All letters and digits may be used more than once. How many passwords are possible?

From 2010, Alberta assigns license plates with 7 characters (ABC-1234), replacing the old 6-character plates (ABC-123). All 26 letters (A to Z) and 10 digits (0 to 9) may be used more than once.

a) How many license plates were possible with the old plate?
b) How many license plates are possible with the new plate?

SOLUTION

Use the fundamental counting principle.

a) For each letter, there are 26 choices.
For each digit there are 10 choices.
So, the number of possible plates is this product:

$$\underset{\substack{\uparrow\\ 1^{\text{st}}\\ \text{letter}}}{\underline{26}} \cdot \underset{\substack{\uparrow\\ 2^{\text{nd}}\\ \text{letter}}}{\underline{26}} \cdot \underset{\substack{\uparrow\\ 3^{\text{rd}}\\ \text{letter}}}{\underline{26}} \cdot \underset{\substack{\uparrow\\ 1^{\text{st}}\\ \text{digit}}}{\underline{10}} \cdot \underset{\substack{\uparrow\\ 2^{\text{nd}}\\ \text{digit}}}{\underline{10}} \cdot \underset{\substack{\uparrow\\ 3^{\text{rd}}\\ \text{digit}}}{\underline{10}} = 17\,576\,000$$

So, 17 576 000 license plates were possible with the old plate.

b) Adding another digit increases the number of plates by a factor of 10:
$17\,576\,000 \cdot 10 = 175\,760\,000$
So, 175 760 000 license plates are possible with the new plate.

THINK FURTHER

In *Example 2*, since the letters I, O, and Q may be mistaken for the numbers 1 and 0, suppose these letters are not used on a plate. How many new plates are possible when the letters I, O, and Q are not used?

The total number of choices may decrease if repetition is not allowed. For example, determine how many 3-digit numbers can be formed using the digits 7, 8, and 9. Consider the number of ways to choose each digit.

Repetition is allowed.
For each digit, there are 3 choices.
Number of ways $= 3 \cdot 3 \cdot 3$
$= 27$

Twenty-seven 3-digit numbers can be formed.

Repetition is not allowed.
There are 3 ways to choose the 1st digit, 2 ways to choose the 2nd digit, and 1 way to choose the 3rd digit.
Number of ways $= 3 \cdot 2 \cdot 1$
$= 6$

Six 3-digit numbers can be formed.

Discuss the Ideas

1. What is the fundamental counting principle?

2. How do you decide whether to use a graphic organizer or the fundamental counting principle to determine the total number of ways to choose objects from one or more sets?

Exercises

A

3. A gift-wrapping booth has 3 sizes of boxes, 2 colours of gift wrap, and 4 different bows. How many choices of wrapping are possible using a box, gift wrap, and bow?

4. A school cafeteria offers a soup and sandwich combo. There are 3 kinds of soup (pea, tomato, black bean) and 4 kinds of sandwiches (egg salad, tuna, veggie, ham). Use a graphic organizer to show the number of possible combos.

5. **a**) How many Alberta license plates were possible in 1912? Assume there were no restrictions on the digits.

b) Suppose 0 is not permitted as the first digit. How many license plates were possible?

B

6. Use the digits 5, 6, 7, and 8.

a) How many 4-digit numbers can be formed when repetition is allowed?

b) How many 4-digit numbers can be formed when repetition is not allowed?

7. How many 2-digit numbers less than 60 are even? Verify your work using another strategy.

8. In the women's gold-medal hockey game at the 2010 Vancouver Olympics, Team Canada defeated Team U.S.A. 2 – 0. Each team had 21 players on its roster. Assume each of the 21 players on Team Canada shook hands with each of the 21 players on Team U.S.A. What was the total number of handshakes?

9. A mobile phone has an 8-digit code called a PIN Unlock Key (PUK). This code is used to unlock a phone when an incorrect PIN number has been used three times in succession. All digits can be used. How many PUK codes are possible?

10. How many radio call letters beginning with C and consisting of 4 letters can be assigned to radio stations in Canada when repetition is not allowed?

11. A pizza chain offers cheese pizzas with a choice of these toppings: pepperoni, onion, sausage, mushrooms, and anchovies.

a) Explain why the pizza chain can claim to offer 32 different pizzas.

b) Another pizza chain claims that with its choice of toppings, it can create just over 1000 pizzas. What is the minimum number of toppings it must offer?

12. There are 700 students in a high school. Explain why at least two students must have the same first initial and the same last initial.

13. Which event is more likely? Why?

- Tossing 23 tails with 23 pennies
- Rolling 9 sixes with 9 dice

C

14. A die has faces labelled 1 to 6. The number of outcomes when n dice are rolled is 279 936. How many dice were rolled? Explain your reasoning.

15. Determine the number of ways of rolling 4 or 7 with two dice labelled from 1 to 6. Explain why you cannot use the fundamental counting principle.

16. Three couples go to see a movie at Cinematheque in Winnipeg. They sit together in 6 consecutive seats and couples sit together. How many seating arrangements are possible?

Multiple-Choice Questions

1. A multiple-choice test has 10 questions. Each question has 4 choices: A, B, C, or D. How many ways can the test be answered?

A. 14 **B.** 40 **C.** 10 000 **D.** 1 048 576

2. How many 4-digit numbers greater than 1000 can be formed with no repetition in their digits?

A. 4536 **B.** 3024 **C.** 9000 **D.** 10 000

3. The final score in a recreational soccer game is 6 – 3. How many scores are possible at the end of the first half?

A. 9 **B.** 10 **C.** 18 **D.** 28

Study Note

When you use the fundamental counting principle to determine the total number of possible choices, why do you multiply instead of add? Use an example to explain.

ANSWERS

Check Your Understanding
1. 27 ways **2.** 67 600 000

Exercises
3. 24 choices **4.** 12 combos **5. a)** 10 000 **b)** 9000 **6. a)** 256 **b)** 24
7. 25 **8.** 441 **9.** 100 000 000 **10.** 13 800 **11. b)** 10
13. Tossing 23 tails with 23 pennies **14.** 7 **15.** 9 ways **16.** 48

Multiple Choice
1. D **2.** A **3.** D

Additional Workspace

8.2 Permutations of Different Objects

FOCUS **Create and apply strategies to determine the number of ways to arrange a set of different objects.**

Get Started

Evaluate each expression without using a calculator.

$\frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$ $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$ $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

Construct Understanding

There are 4 ferry routes from the Vancouver area to Vancouver Island.
In how many ways is it possible to travel to Vancouver Island by one ferry route and return to the Vancouver area by a different route?
How would your answer change if a 5th route was added?
A 6th route?
What patterns do you see?

An arrangement of a set of objects is called a **permutation**.
The word *permutation* comes from the Latin words *per* + *mutare* that together mean “by change” or “through change.”
In a permutation, order matters; for example, a pin code for a debit card.

To determine the number of 7-letter permutations of KELOWNA, use the fundamental counting principle.

There are 7 ways to choose the first letter,
6 ways to choose the second letter,
5 ways to choose the third letter,
4 ways to choose the fourth letter,
3 ways to choose the fifth letter,
2 ways to choose the sixth letter,
and 1 way to choose the last letter.
$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$
There are 5040 ways to arrange the letters in KELOWNA.

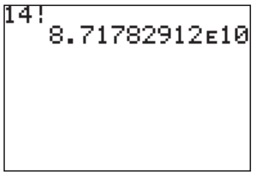
The expression $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be represented as 7!
This is **factorial notation.** 7! is read as seven factorial.
The factorial sign, !, means to take the product of all natural numbers less than or equal to the given number.

For example,
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
In general, for any natural number n:

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$
$$0! = 1$$

$n!$ represents the number of permutations of n different objects.

To evaluate a factorial on a TI-83 Plus or TI-84 graphing calculator:
Enter the number, press MATH ◄ to select PRB, then press 4 ENTER.
When $n \geq 14$, $n!$ is very large and most calculators display an approximate number.



Example 1 Determining the Number of Permutations of *n* Different Objects

Check Your Understanding

1. A puzzle designer decides to scramble the letters in the word EDUCATION to create a jumble puzzle. How many 9-letter permutations of EDUCATION can be created?

A student has a jumble puzzle phone app. He is trying to identify a common word that has been scrambled as LOUVME. How many 6-letter permutations of LOUVME can be created?

SOLUTION

The number of permutations is: $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 720$

There are 720 six-letter permutations of LOUVME.

Only *some* of the objects from a set may be arranged.
That is, the number of permutations of a set of objects chosen from a larger set can be determined. For example, Chico has 7 songs on his iPod. He has time to listen to 3 songs on his way to school. How many arrangements of 3 different songs are possible?

There are 7 choices for the first song, 6 choices for the second song, and 5 choices for the third song. So, the number of arrangements of 3 songs is: $7 \cdot 6 \cdot 5 = 210$
This product can be represented using factorial notation.

$$\begin{aligned} 7 \cdot 6 \cdot 5 &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{7!}{4!}, \text{ or } \frac{7!}{(7-3)!} \\ &= 210 \end{aligned}$$

7 is the number of songs from which he chooses.

3 is the number of songs he listens to.

The notation $_7P_3$ represents the number of ways of choosing 3 from 7.

This relationship can be expressed in general terms.

Permutations of Different Objects

The number of permutations of n distinct objects taken r at a time is:

$${}_nP_r = \frac{n!}{(n-r)!}, n \geq r$$

Suppose Chico has 3 songs on his iPod and he wants to listen to all 3 songs.
There are $3 \cdot 2 \cdot 1$, or 6 arrangements of the 3 songs.
This number of arrangements can also be determined by using the formula:

$$ {}_nP_r = \frac{n!}{(n-r)!} \quad \text{Substitute: } n = 3 \text{ and } r = 3$$

$$ {}_3P_3 = \frac{3!}{(3-3)!}$$

$$ = \frac{3!}{0!}$$

$$ = \frac{6}{0!}$$

Compare the two results. Since $\frac{6}{0!}$ must equal 6, 0! is equal to 1.

THINK FURTHER

What are the meanings and values of ${}_nP_0$ and ${}_nP_1$?

Example 2 — Determining the Number of Permutations of *n* Different Objects Taken *r* at a Time

How many ways can a president, vice-president, and secretary-treasurer for a high school Safe Grad committee be selected from 58 Grade 12 students?

SOLUTION

Choosing a particular student for president is different from choosing that student for vice-president or secretary-treasurer. That is, order matters. So, use the permutation formula.

$$ {}_nP_r = \frac{n!}{(n-r)!} \quad \text{Substitute: } n = 58 \text{ and } r = 3$$

$$ {}_{58}P_3 = \frac{58!}{(58-3)!}$$

$$ = \frac{58!}{55!}$$

$$ = 58 \cdot 57 \cdot 56$$

$$ = 185\,136$$

There are 185 136 ways to select the Safe Grad committee.

Check Your Understanding

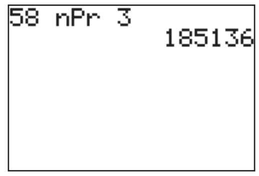
2. Eight students are competing in a 200-m race. How many ways can the students finish first, second, and third?

THINK FURTHER

In *Example 2*, suppose students were not assigned to positions of president, vice-president, and secretary-treasurer. Would the number of ways to select the committee increase or decrease? Explain.

A graphing calculator can be used to evaluate a permutation.

To evaluate ${}_{58}P_3$ on a TI-83 Plus or TI-84 graphing calculator: Enter 58, press MATH ◄ to select PRB, then press 2 3 ENTER.



Example 3 Solving an Equation Involving Permutations

Check Your Understanding

3. Solve each equation for n or r.

a) ${}_nP_2 = 56$ **b)** ${}_5P_r = 20$

Solve each equation for n or r.

a) ${}_nP_3 = 60$ **b)** ${}_6P_r = 30$

SOLUTION

a) ${}_nP_3 = 60$

Use the permutation formula.

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{Substitute: } {}_nP_r = 60, r = 3$$

$$60 = \frac{n!}{(n-3)!}$$

$$60 = \frac{n(n-1)(n-2)\cancel{(n-3)}\cancel{(n-4)}\ldots}{\cancel{(n-3)}\cancel{(n-4)}\ldots}$$

$$60 = n(n-1)(n-2)$$

Use logical reasoning to find 3 consecutive numbers whose product is 60. The perfect cube, 64, is close to 60.

$64 = 4 \cdot 4 \cdot 4$

So, try 3 consecutive numbers with 4 as the middle number:

$5 \cdot 4 \cdot 3$

$5 \cdot 4 \cdot 3 = 60$

So, $n = 5$

b) ${}_6P_r = 30$

Use the permutation formula.

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{Substitute: } {}_nP_r = 30, n = 6$$

$$30 = \frac{6!}{(6-r)!}$$

$$30 = \frac{720}{(6-r)!}$$

$$(6-r)! = \frac{720}{30}$$

$$(6-r)! = 24$$

Since $4! = 24$, then

$6 - r = 4$

$r = 2$

In *Example 2*, an equation involving permutations can be used to verify the solution.

Use the equation ${}_{n}P_{r} = \frac{n!}{(n-r)!}$.

Substitute ${}_{n}P_{r} = 185\,136$ and $r = 3$, then solve for n.

$$185\,136 = \frac{n!}{(n-3)!}$$

$$185\,136 = n(n-1)(n-2)$$

Look for 3 consecutive numbers whose product is 185 136.
Use a calculator.
$\sqrt[3]{185\,136} \doteq 57$
So, try 3 consecutive numbers with 57 as the middle number: $58 \cdot 57 \cdot 56$
$58 \cdot 57 \cdot 56 = 185\,136$
So, $n = 58$
Since there were 58 Grade 12 students, the solution is correct.

Discuss the Ideas

1. In the permutation formula, ${}_{n}P_{r} = \frac{n!}{(n-r)!}$, $n \geq r$, explain why the n objects must be different.

2. In the notation ${}_{n}P_{r}$, why must n be greater than or equal to r?

Exercises

A

3. Evaluate each factorial.

a) 4! **b)** 1! **c)** 5! **d)** 0!

4. Determine each value.

a) ${}_7P_2$ **b)** ${}_5P_5$

c) ${}_8P_7$ **d)** ${}_6P_1$

5. a) Use a graphic organizer to determine the number of ways to arrange the letters in each word.

i) ELK **ii)** LYNX

b) Use factorial notation to determine the number of ways to arrange the letters in each word.

i) BISON **ii)** FALCON

B

6. **a)** How many 2-letter permutations are there for each word in question 5?

b) How many 3-letter permutations are there for each word in question 5?

c) Describe any patterns you notice.

7. The music teacher must arrange 5 tunes for the senior jazz band to perform at Music Night. She has 20 tunes to choose from.

a) How many arrangements are possible?

b) What other strategy could you use to determine the number of arrangements?

8. In the World Cup of soccer, 32 teams compete for the title. What is the number of ways that the winner, runners-up, third, and fourth place prizes could be awarded? Verify your answer.

9. The longest English non-technical words with no repeated letters are *dermatoglyphics*, *misconjugatedly*, and *uncopyrightable*. What is the total number of ways to arrange all the letters in each word?

10. Solve each equation for n or r.

a) ${}_nP_2 = 90$ **b)** ${}_nP_3 = 120$

c) ${}_{9}P_{r} = 72$ d) ${}_{7}P_{r} = 210$

C

11. A student has 15 video games: 4 adventure games, 4 arcade games, 2 puzzle games, and 5 simulation games. How many ways can the games be positioned on a shelf if the games stay with their genre?

12. a) Seven different keys are to be placed on a key ring. How many ways can the keys be arranged?

b) How many ways can n distinct keys be arranged on a key ring? Explain.

Multiple-Choice Questions

1. How many ways can all the letters in the word ROCKIES be arranged?

   A. 5040 **B.** 49 **C.** 16 807 **D.** 117 649

2. Which expression cannot be evaluated?

   A. ${}_8P_5$ **B.** ${}_8P_0$ **C.** ${}_8P_8$ **D.** ${}_5P_8$

3. There are 8 swimmers in the finals of the 100-m butterfly. How many ways could the gold, silver, and bronze medals be awarded?

   A. 24 **B.** 336 **C.** 512 **D.** 40 320

Study Note

Explain what the notation ${}_4P_2$ represents. Write a problem that can be solved using ${}_4P_2$, then solve the problem.

ANSWERS

Check Your Understanding

1. 362 880 **2.** 336 ways **3. a)** $n = 8$ **b)** $r = 2$

Exercises

3. a) 24 **b)** 1 **c)** 120 **d)** 1 **4. a)** 42 **b)** 120 **c)** 40 320 **d)** 6 **5. a) i)** 6 **ii)** 24 **b) i)** 120 **ii)** 720 **6. a)** 6; 12; 20; 30 **b)** 6; 24; 60; 120 **7. a), b)** 1 860 480 arrangements **8.** 863 040 ways **9.** approximately 1.31×10^{12} ways **10. a)** $n = 10$ **b)** $n = 6$ **c)** $r = 2$ **d)** $r = 3$ **11.** 3 317 760 ways **12. a)** 720 ways **b)** $(n - 1)!$ ways

Multiple Choice

1. A **2.** D **3.** B

Additional Workspace

8.3 Permutations Involving Identical Objects

FOCUS **Determine the number of arrangements from a set containing identical objects.**

Get Started

Determine each value.

${}_3P_0$ ${}_3P_1$ ${}_3P_2$ ${}_3P_3$

Explain why ${}_3P_2$ and ${}_3P_3$ are equal.

Construct Understanding

Dave, Ella, and Anna listed the 4-letter permutations of the letters in their names. Which name has the greatest number of permutations? How does the number of identical letters affect the number of permutations?

In Lesson 8.2, the permutation problems involved objects that were different. Sometimes some of the objects are identical.
Consider the letters in the word DEER. If all the letters were different, the number of arrangements would be: $4! = 24$
However, each word contains 2 Es. In each word, when the Es are interchanged, the result is the same word.
$DE_1E_2R \qquad DE_2E_1R$
There are 2! ways of arranging E_1 and E_2 in a word.
So, the number of arrangements becomes: $\frac{4!}{2!} = 12$
Similarly, if all the letters in the word GEESE were different, the number of arrangements would be: $5! = 120$
However, each word contains 3 Es. In each word, when the Es are interchanged, the result is the same word.
$GE_1E_2SE_3 \quad GE_1E_3SE_2 \quad GE_2E_1SE_3 \quad GE_2E_3SE_1 \quad GE_3E_2SE_1 \quad GE_3E_1SE_2$
There are 3! ways of arranging E_1, E_2, and E_3 in a word.
So, the number of arrangements is: $\frac{5!}{3!} = 20$
Similarly, if all the letters in the word PEEWEE were different, the number of arrangement would be: $6! = 720$
However, each word contains 4 Es. There are 4! ways of arranging E_1, E_2, E_3, and E_4 in a word. So, the number of arrangements is: $\frac{6!}{4!} = 30$

Word is used to mean an arrangement of letters.

THINK FURTHER

Explain why the number of ways to arrange the letters in HOODOO is the same as the number of ways to arrange the letters in PEEWEE?

Permutations of Identical Objects

The number of permutations of n objects with r identical objects is: $\frac{n!}{r!}$

Example 1 Determining the Number of Permutations of Some Identical Objects

There are 8 cans of soup on a shelf. Three of the cans contain tomato soup and the other 5 cans are all different. How many ways can the cans be arranged in a row?

SOLUTION

There are 8 cans of soup.
Three of the cans are identical.
The number of ways the cans can be arranged in a row is:

$$\frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$
$$= 6720$$

The cans can be arranged in a row in 6720 ways.

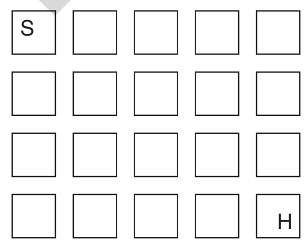
Check Your Understanding

1. There are 7 boxes of cereal on a shelf. Five of the boxes are bran cereal, one box is puffed wheat, and the other box is granola. How many ways can the boxes be arranged in a row?

Consider a collection of 3 identical soccer balls, 2 identical baseballs, and 1 basketball. If all the sports balls were different, they could be arranged in a row in 6!, or 720 ways.
For each arrangement of the sports balls, determine the number of equivalent arrangements.
There are 3 soccer balls, so there are 3!, or 6 ways to arrange the soccer balls without a visible difference.
There are 2 baseballs, so there are 2!, or 2 ways to arrange the baseballs without a visible difference.
So, the number of ways to arrange the sports balls is: $\frac{6!}{3!2!1!} = 60$
When there are identical objects in a set of objects, the number of permutations in the set is less than the number of permutations with distinct objects.

Example 2 Determining the Number of Permutations of Objects of Two Kinds

Abby walks 9 blocks from her home, H, to school, S. She always walks 5 blocks west and 4 blocks north. How many ways can Abby walk to school?



SOLUTION

The total number of blocks walked is 9.
There are two kinds of blocks.
Blocks walked west: 5
Blocks walked north: 4
The number of ways Abby can walk to school is:

$$\begin{aligned}\frac{9!}{5!4!} &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{4!}\\ &= \frac{9 \cdot {}^{2}\cancel{8} \cdot 7 \cdot {}^{\cancel{3}}\cancel{6}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}\\ &= 126\end{aligned}$$

Abby can walk to school in 126 ways.

Check Your Understanding

2. Graeme walks 8 blocks from his home to the library. He always walks 4 blocks east and 4 blocks south. How many ways can Graeme walk to the library?

THINK FURTHER

How many permutations of n identical objects are there? Explain your thinking.

Permutations of Objects of Multiple Kinds

In a set of n objects with n_1 objects of one kind, n_2 objects of another kind, n_3 objects of another kind, and so on, for k kinds of objects, the number of permutations is:

$$\frac{n!}{n_1!n_2!n_3!\ldots n_k!}, \text{ where } n_1 + n_2 + n_3 + \ldots + n_k = n$$

Example 3 Determining the Number of Permutations of Objects of Multiple Kinds

There are 12 pieces of trail mix in a dish. There are 5 almonds, 4 dried banana slices, and 3 papaya chunks. Corey eats the trail mix one piece at a time. In how many ways can Corey eat all the trail mix?

SOLUTION

The total number of pieces is 12.
There are three kinds of pieces: almonds, dried banana slices, and papaya chunks
Number of almonds: 5
Number of dried banana slices: 4
Number of papaya chunks: 3
The number of ways Corey can eat the trail mix is:
$\frac{12!}{5!4!3!} = 27\ 720$
Corey can eat the trail mix in 27 720 ways.

Check Your Understanding

3. A kabob recipe requires 2 mushrooms, 2 shrimp, 2 cherry tomatoes, and 2 zucchini slices. How many ways can Amelie arrange these items on a skewer?

Discuss the Ideas

1. How is the number of permutations of a set of n distinct objects related to the number of permutations of a set of n objects with r identical objects?

2. Explain when you would use each expression to determine the number of permutations of a set of n objects:

$n!, \frac{n!}{(n-r)!}, \frac{n!}{r!}$, and $\frac{n!}{n_1!n_2!n_3!\cdots n_k!}$

Exercises

A

3. Evaluate.

a) $\frac{4!}{2!2!}$

b) $\frac{9!}{3!6!}$

c) $\frac{8!}{2!3!4!}$

d) $\frac{10!}{2!2!5!}$

4. Which word in each pair has the greater number of permutations of all its letters?

a) BID or BIB

b) DEED or DIED

c) KAYAK or KOALA

d) RUDDER or REDDER

B

5. How many permutations are there of the 4 digits in each number?

a) 1234

b) 1123

c) 1113 **d**) 1111

6. **a**) How many permutations are there of all the letters in each of these Aboriginal words?

i) ISKWEW **ii**) TSILIKST

iii) SUMSHASAT **iv**) KINNIKINNICK

b) How do identical letters change the number of permutations?

7. The number of permutations of all the letters in the word BRICK is 120. How can you use this information to determine the number of permutations of all the letters in the word BROOK?

8. The number of permutations of all the digits in a 5-digit number is one. What do you know about the number? Justify your answer.

9. How many 9-digit numbers can be created from the digits 5, 5, 6, 6, 6, 7, 7, 7, 7?

10. Create a 5-digit number so that the number of permutations of all the digits is:

a) the greatest possible

b) the least possible

c) 10

d) 5

11. Identify a common word that satisfies each requirement.

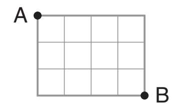
a) Contains 3 letters; numbers of permutations of all letters is 6.

b) Contains 4 letters; numbers of permutations of all letters is 12.

c) Contains 4 letters; numbers of permutations of all letters is 24.

d) Contains 5 letters; numbers of permutations of all letters is 30.

12. **a**) How many ways are there to get from A to B travelling along grid lines and moving only to the right or down?



b) Why does order matter in this problem?

C

13. How many ways can all the letters in the word ABACUS be arranged so that the vowels are always together?

14. A 26-term series is written using only the integers +8 and −5. How many such series can be written with a sum of 0? Explain your reasoning.

Multiple-Choice Questions

1. What is the number of permutations of the letters in XWAYXWAY, an ancient Aboriginal village previously located in what is now Vancouver's Stanley Park?

A. 70 **B.** 1680 **C.** 2520 **D.** 40 320

2. Which expression can be used to determine the number of permutations of the digits in the number 111 122?

A. $6!$ **B.** $\frac{6!}{2!}$ **C.** $\frac{6!}{4!2!}$ **D.** $\frac{6!}{4!}$

3. A coin is tossed 12 times. What is the number of ways the coin can land with 6 heads and 6 tails?

A. 924 **B.** 665 280 **C.** 479 001 600 **D.** 13 305 600

Study Note

Explain why there are fewer permutations of a set of objects when some of the objects are identical than when all the objects are different. Include examples in your explanation.

ANSWERS

Check Your Understanding

1. 42 ways **2.** 70 ways **3.** 2520 ways

Exercises

3. a) 6 **b)** 84 **c)** 140 **d)** 7560 **4. a)** BID **b)** DIED **c)** KOALA **d)** RUDDER **5. a)** 24 **b)** 12 **c)** 4 **d)** 1 **6. a) i)** 360 **ii)** 5040 **iii)** 30 240 **iv)** 138 600 **7.** 60 **9.** 1260 **12. a)** 35 ways **13.** 72 ways **14.** 5 311 735 series

Multiple Choice

1. C **2.** C **3.** A

Additional Workspace

CHECKPOINT

Self-Assess

Can you . . .	Try *Checkpoint* question	For review, see
use a graphic organizer to count the number of possible choices that can be made?	1	Page 3 in Lesson 8.1 (Example 1)
use the fundamental counting principle to solve a problem?	3	Page 4 in Lesson 8.1 (Example 2)
count the number of ways objects can be arranged in a row?	4	Page 13 in Lesson 8.2
use factorial notation to determine the number of permutations of *n* different objects taken *n* at a time?	5	Page 14 in Lesson 8.2 (Example 1)
use a variety of strategies to determine the number of permutations of *n* different objects taken *r* at a time?	6	Page 15 in Lesson 8.2 (Example 2)
solve an equation that involves ${}_nP_r$ notation?	7	Page 16 in Lesson 8.2 (Example 3)
determine the number of permutations when two or more objects are identical?	9	Page 27 in Lesson 8.3 (Example 3)

Assess Your Understanding

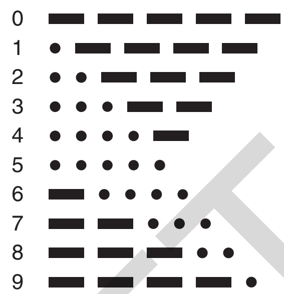
8.1

1. A garage door remote has 10 code switches. Each switch can be positioned up or down to create a wireless code. How many codes are possible?

2. **Multiple Choice** A restaurant offers a meal combo that consists of a beverage, a main course, and a dessert. There are 5 beverages, 6 main courses, and 4 desserts. How many meal combos are available?

A. 15 **B.** 30 **C.** 20 **D.** 120

3. Morse code uses arrangements of 5 characters to represent the digits 0 through 9. Each character is either a dot or a dash. How many arrangements of 5 characters are possible?



8.2

4. **Multiple Choice** How many 5-letter permutations of YUKON can be created?

A. 6 **B.** 24 **C.** 120 **D.** 3125

5. A family of six is to be seated in a row for a photo. The mother and father must be at either end. How many ways can the family be arranged?

6. An under-10 house-league soccer team has 11 players. Seven players are on the field at a time. How many ways can 7 starters be chosen from the members of the team?

7. Solve each equation for n or r.

a) ${}_{n}P_{2} = 42$

b) ${}_{7}P_{r} = 840$

8.3

8. **Multiple Choice** How many ways can 2 pennies, 3 nickels, and 5 quarters be arranged in a row?

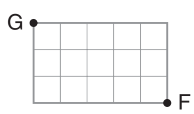
A. 30 **B.** 2520 **C.** 5040 **D.** 3 628 800

9. What is the number of permutations of all the letters in the name of each provincial park?

a) VERMILION

b) OPAPISKAW

10. How many ways are there to get from F to G travelling along grid lines and moving only to the left or up?



ANSWERS

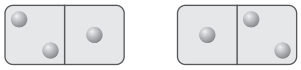
1. 1024 **2.** D **3.** 32 **4.** C **5.** 48 ways **6.** 1 663 200 ways **7. a)** $n = 7$ **b)** $r = 4$ **8.** B **9. a)** 181 440 **b)** 90 720 **10.** 56

8.4 Combinations

FOCUS **Develop and apply strategies to determine the number of combinations of a set of different objects.**

Get Started

The game Mexican Train uses a set of double-twelve domino tiles.



2–1 domino 1–2 domino

Each tile is rectangular with a centre line dividing the domino face into two ends. Each end is marked with a number of dots or "pips", ranging from 0 to 12. How many different dominoes are possible?
Assume a 2–1 domino is the same as a 1–2 domino.

Construct Understanding

Solve each problem.

A. At a restaurant, each meal comes with a choice of 4 side dishes. A regular diner is asked to rank her 2 favourite side dishes in order of preference. How many different rankings are possible?

B. At another restaurant, a diner can choose 2 of 4 side dishes to accompany a meal. In how many ways can a diner choose 2 side dishes?

How are the problems similar? How are they different?

A selection of objects where order does not matter is a **combination**. Consider the number of ways a reader can choose 3 books from 4 books. Unlike positioning books on a library shelf, order does not matter. Choosing books 1, 2, and 3 is the same as choosing books 2, 1, and 3, and so on.

Combinations of 3 books	**Permutations of each combination**
123	123, 132, 213, 231, 312, 321
124	124, 142, 241, 214, 412, 421
134	134, 143, 341, 314, 413, 431
234	234, 243, 342, 324, 423, 432
4 combinations	24 permutations

When order matters, there are ${}_4P_3$, or 24 ways to choose 3 books from 4 books. There are 3!, or 6 ways to choose the same 3 books.

So, the number of combinations is: $\frac{24}{3!} = 4$

In general, r objects can be chosen from n different objects in ${}_nP_r$ ways, and r objects can be arranged in $r!$ ways. So, the number of combinations of n different objects taken r at a time is:

$${}_nC_r = \frac{{}_nP_r}{r!} \quad \text{Substitute: } {}_nP_r = \frac{n!}{(n - r)!}$$
$$= \frac{n!}{(n - r)!r!}$$

Combinations of Different Objects

The number of combinations of n distinct objects taken r at a time is:

$${}_nC_r = \frac{n!}{(n - r)!r!},\ n \geq r$$

${}_nC_r$ can also be written using the notation $\binom{n}{r}$; it is read "n choose r."

Example 1 Determining the Number of Combinations of *n* Different Objects Taken *r* at a Time

Lotto Max is a Canadian lottery where a player chooses 7 numbers from 1 to 49. To win the jackpot, all 7 numbers must match the drawn numbers. How many combinations of 7 numbers are possible?

SOLUTION

The order in which the numbers are drawn does not matter. So, use the combination formula.

$${}_nC_r = \frac{n!}{(n - r)!r!} \qquad \text{Substitute: } n = 49 \text{ and } r = 7$$
$${}_{49}C_7 = \frac{49!}{(49 - 7)!7!}$$
$$= \frac{49!}{42!7!}$$
$$= \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 85\ 900\ 584$$

There are 85 900 584 possible combinations of 7 numbers.

Check Your Understanding

1. In the Keno lottery, 20 numbers from 1 to 80 are chosen. How many combinations of 20 numbers are possible?

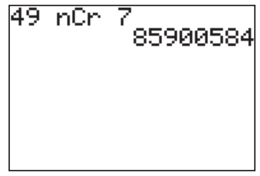
THINK FURTHER

In *Example 1*, how can you use the calculations for Lotto Max to determine the number of combinations for Lotto 6/49, where 6 numbers are chosen from 49?

A graphing calculator can be used to evaluate a combination.

To evaluate ${}_{49}C_7$ on a TI-83 Plus or TI-84 graphing calculator:

Enter 49, press MATH ◄ to select PRB, then press 3 7 ENTER.



Check Your Understanding

2. A local arena has 10 applicants interested in working in the snack bar.

a) How many ways can 4 applicants be chosen?

b) How many ways can 6 applicants be chosen?

Example 2 Relating Combinations

A fund-raising committee is to be chosen from a group of 8 students.

a) How many ways can a committee of 5 students be chosen?

b) How many ways can a committee of 3 students be chosen?

SOLUTION

The order in which the students are chosen does not matter. So, use the combination formula.

a) $${}_nC_r = \frac{n!}{(n - r)!r!}$$ Substitute: $n = 8$ and $r = 5$

$${}_8C_5 = \frac{8!}{(8 - 5)!5!}$$
$$= \frac{8!}{3!5!}$$
$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$$
$$= 56$$

A committee of 5 students can be chosen in 56 ways.

b) $${}_nC_r = \frac{n!}{(n - r)!r!}$$ Substitute: $n = 8$ and $r = 3$

$${}_8C_3 = \frac{8!}{(8 - 3)!3!}$$
$$= \frac{8!}{5!3!}$$
$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$$
$$= 56$$

A committee of 3 students can be chosen in 56 ways.

In *Example 2*, the answers to parts a and b are the same. This is because the number of ways of choosing 5 students from 8 students is the same as the number of ways of choosing 3 students (that is, not choosing 5 students) from 8 students.

This relationship can be expressed in general terms.

$$_nC_r = \frac{n!}{(n-r)!r!} \qquad \text{Replace } r \text{ with } (n-r).$$

$$\begin{aligned} _nC_{n-r} &= \frac{n!}{(n-(n-r))!(n-r)!} \\ &= \frac{n!}{(n-n+r)!(n-r)!} \\ &= \frac{n!}{r!(n-r)} \\ &= {}_nC_r \end{aligned}$$

The number of ways of choosing r objects from a set of n objects is the same as the number of ways of not choosing r objects from a set of n objects. So,

$$_nC_r = {}_nC_{n-r} \text{ or}$$

$$\binom{n}{r} = \binom{n}{n-r},\ n \geq r$$

THINK FURTHER

Does $_nP_r = {}_nP_{n-r}$? Explain.

Example 3 Solving a Problem That Involves More than One Combination

One year, the minimum admission requirements for the University of British Columbia are a secondary school graduation diploma and successful completion of these Grades 11 and 12 courses.

- One of English 12 or English 12 First Peoples
- Three of 16 approved examinable Grade 12 courses
- English 11
- One of 3 math courses
- Civic Studies 11 or Social Studies 11
- One of 5 approved Science courses
- One of 24 approved Language courses

How many ways can a student meet the minimum admission requirements?

Check Your Understanding

3. A new store must hire 3 cashiers and 4 stock clerks. There are 7 applicants for cashier and 8 applicants for stock clerk. How many ways can the 7 employees be chosen?

SOLUTION

The minimum requirements include the successful completion of 9 specific courses.
The order in which the courses are taken does not matter.

Requirement	Number of Choices
1 Grade 12 English course	2
3 approved examinable Grade 12 courses	${}_{16}C_3 = 560$
English 11	1
1 math course	3
Civic Studies 11 or Social Studies 11	2
1 Science course	5
1 Language course	24

Use the fundamental counting principle to determine the number of ways a student can meet the minimum requirements:
$2 \cdot 560 \cdot 1 \cdot 3 \cdot 2 \cdot 5 \cdot 24 = 806\,400$
There are 806 400 ways to meet the admission requirements to U.B.C.

Discuss the Ideas

1. How do you recognize a combination problem?

2. How would you use mental math to determine ${}_{10}C_2$?

3. For what values of n and r is the formula ${}_nC_r = \frac{n!}{(n - r)!r!}$ defined? Justify the answer.

Exercises

A

4. Evaluate.

a) $\frac{10!}{3!7!}$

b) $\frac{6!}{1!5!}$

c) $\frac{12!}{3!9!}$

d) $\frac{15!}{13!2!}$

5. Determine each value.

a) ${}_4C_2$

b) $\binom{7}{5}$

c) ${}_{10}C_{10}$

d) $\binom{12}{8}$

6. How many combinations of each number of letters can be formed from the letters in the word LINE? List the combinations each time.

a) 1

b) 2

c) 3 **d)** 4

B

7. These are the names of lakes in western Canada. How many 4-letter combinations can be formed using the letters in each name?

a) BISTCHO **b)** TOEWS

c) HOIDAS **d)** COQUITLAM

8. a) How many 3-digit permutations are there of the digits in the number 67 512?

b) How can you use your answer to part a to determine how many 3-digit combinations are possible?

9. Would you use a permutation or a combination to represent each situation? Justify your choice.

a) choosing 3 out of 4 musical notes to create a tune

b) choosing 3 out of 4 sweatshirts to take camping

c) choosing 3 out of 4 contestants to advance to the next round

d) choosing 3 out of 4 digits to create a password

10. Rafael has a list of his mom's 15 favourite songs. He will download 7 of these songs to her iPod.

a) How many ways can Rafael select 7 songs to download?

b) Suppose Rafael downloads 8 songs. Without doing any calculations, how many ways can he select 8 songs? Explain your strategy.

11. At the Soccer World Cup, 16 of the 32 teams advance beyond the second round. How many ways can 16 teams advance? Did you use a permutation or a combination to solve this problem? Explain.

12. When Tanner's team won the final game in the Genesis Hospitality High School hockey tournament in Brandon, Manitoba, each of the 6 players on the ice gave each other a high five. How many high fives were there?

13. A test has 2 parts. Students must answer 10 of 15 questions from part A and write 3 essays from a choice of 5 essay topics in part B. What is the number of possible responses to the test?

14. A jury of 6 men and 6 women is to be chosen from a jury pool of 12 men and 15 women. How many juries are possible?

15. Solve each equation for n or r.

a) ${}_nC_2 = 28$

b) ${}_nC_4 = 35$

c) ${}_4C_r = 6$

d) ${}_6C_r = 20$

C

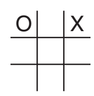
16. From a standard deck of 52 playing cards, how many ways can each hand of 5 cards be dealt?

a) any 5 cards

b) 5 black cards

c) exactly 2 diamonds

17. Two players take turns writing X and O in a 3-by-3 grid until all the cells are full. How many ways are there to fill all the cells with Xs and Os?



Multiple-Choice Questions

1. Which expression is *not* equivalent to ${}_7C_6$?

A. $\frac{7!}{6!(7 - 6)!}$ **B.** ${}_7C_1$ **C.** $\frac{{}_7P_6}{6!}$ **D.** $\binom{6}{7}$

2. Twelve points are located on the circumference of a circle. Lines are drawn to connect all possible pairs of points. How many lines are drawn?

A. 24 **B.** 66 **C.** 132 **D.** 144

3. How many committees of 3 men and 7 women can be selected from a group of 8 men and 10 women?

A. ${}_8C_3 \cdot {}_{10}C_7$ **B.** ${}_8P_3 \cdot {}_{10}P_7$ **C.** ${}_{18}C_{10}$ **D.** ${}_8C_3 + {}_{10}C_7$

Study Note

What is the difference between a permutation and a combination? Use an example to explain.

ANSWERS

Check Your Understanding

1. approximately 3.54×10^{18} combinations **2. a)** 210 ways **b)** 210 ways
3. 2450 ways

Exercises

4. a) 120 **b)** 6 **c)** 220 **d)** 105 **5. a)** 6 **b)** 21 **c)** 1 **d)** 495 **6. a)** 4: L, I, N, E **b)** 6: LI, LN, LE, IN, IE, NE **c)** 4: LIN, LIE, INE, LNE **d)** 1: LINE **7. a)** 35 **b)** 5 **c)** 15 **d)** 126 **8. a)** 60 **b)** 10 **9. a)** permutation **b)** combination **c)** combination **d)** permutation **10. a)** 6435 ways **b)** 6435 ways **11.** 601 080 390 ways; combination **12.** 15 **13.** 30 030 responses **14.** 4 624 620 juries **15. a)** $n = 8$ **b)** $n = 7$ **c)** $r = 2$ **d)** $r = 3$ **16. a)** 2 598 960 ways **b)** 65 780 ways **c)** 712 842 ways **17.** 252 ways

Multiple Choice

1. D **2.** B **3.** A

Additional Workspace

MATH LAB

8.5 Pascal's Triangle

FOCUS **Understand and apply patterns and relationships in Pascal's triangle.**

Get Started

Evaluate.

${}_2C_0$ ${}_3C_1$ ${}_5C_2$

Construct Understanding

A. This triangle is called ***Pascal's triangle***. Use the patterns in the triangle to complete rows 6 to 8. Describe characteristics of the triangle.

1	Row 1
1 1	Row 2
1 2 1	Row 3
1 3 3 1	Row 4
1 4 6 4 1	Row 5
1 $\square$ 10 10 $\square$ 1	Row 6
1 6 $\square$ 20 $\square$ 6 1	Row 7
$\square$	Row 8

B. How can you determine the numbers in row 9 of the triangle? Write the numbers in row 9.

C. Evaluate each combination.

${}_4C_0$ ${}_4C_1$ ${}_4C_2$

${}_4C_3$ ${}_4C_4$

How are the results related to Pascal's triangle?

Choose a value of n between 2 and 7.
Evaluate ${}_nC_r$ for r from 0 to your chosen n.
How are the results related to a row in Pascal's triangle?

D. Explain why the first two rows of Pascal's triangle can be written as shown. Write rows 3 to 5 using ${}_nC_r$ notation.

$${}_0C_0$$

$${}_1C_0 \quad {}_1C_1$$

$$\square \quad \square \quad \square$$

$$\square \quad \square \quad \square \quad \square$$

$$\square \quad \square \quad \square \quad \square \quad \square$$

E. What is the relationship between the row number and the value of n in ${}_nC_r$?

What is the relationship between the rth number in row n and the value of r in ${}_nC_r$?

Use your results to describe the location of ${}_{11}C_8$ in Pascal's triangle.
What is its value?

Assess Your Understanding

1. These are the numbers in row 10 of Pascal's triangle. Use these numbers to generate the numbers in row 11.

1 9 36 84 126 126 84 36 9 1

2. Use the completed Pascal's triangle on page 51 to evaluate each expression. Use a calculator to verify your answers.

a) ${}_4C_1$ **b)** ${}_7C_2$

c) ${}_6C_3$ **d)** ${}_5C_0$

3. Use combinations to determine the numbers in row 13 of Pascal's triangle.

4. Determine the value of each number in Pascal's triangle.

a) the second number in row 20

b) the fourth number in row 24

ANSWERS

1. 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
2. a) 4 **b)** 21 **c)** 20 **d)** 1 **3.** 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1
4. a) 19 **b)** 1771

8.6 The Binomial Theorem

FOCUS Use number patterns and combinations to expand binomials.

Get Started

These are the terms in a row of Pascal's triangle.
1 11 55 165 330 462 462 330 165 55 11 1
In which row are these terms? How do you know?
Write the fifth term in the row using ${}_nC_r$ notation.

Each number in Pascal's triangle is a term in a row.

Construct Understanding

Expand and simplify each power of the binomial $(x + y)$.
$(x + y)^0$ $(x + y)^1$ $(x + y)^2$ $(x + y)^3$ $(x + y)^4$
Describe any patterns.
Use the patterns to predict the expansion of $(x + y)^5$.
Expand to check your prediction.

Compare the expansion of the binomial power $(x + y)^n$ with Pascal's triangle.

$(x + y)^0 = 1$
$(x + y)^1 = 1x + 1y$
$(x + y)^2 = 1x^2 + 2xy + 1y^2$
$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$
$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

1	Row 1
1 1	Row 2
1 2 1	Row 3
1 3 3 1	Row 4
1 4 6 4 1	Row 5

The coefficients of the expansion of $(x + y)^n$ are the terms of row $(n + 1)$ of Pascal's triangle.
The first power of x is x^n and the exponent of the power decreases by 1 for each subsequent term, with the last power being x^0. The first power of y is y^0 and the exponent of the power increases by 1 for each subsequent term, with the last power being y^n. The sum of the exponents in each term is n.

Example 1 Using Pascal's Triangle and Patterns to Expand

Expand, then simplify $(2a - 3)^5$.

SOLUTION

The exponent is 5, so list the terms in row 6 of Pascal's triangle.
1, 5, 10, 10, 5, 1
These are the coefficients of the terms in the expansion of the binomial:
$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$
Substitute $x = 2a$ and $y = -3$, then simplify.

$$\begin{aligned}(2a - 3)^5 &= 1(2a)^5 + 5(2a)^4(-3) + 10(2a)^3(-3)^2 + 10(2a)^2(-3)^3 \\ &\quad + 5(2a)(-3)^4 + 1(-3)^5 \\ &= 32a^5 + 5(16a^4)(-3) + 10(8a^3)(9) + 10(4a^2)(-27) \\ &\quad + 5(2a)(81) - 243 \\ &= 32a^5 - 240a^4 + 720a^3 - 1080a^2 + 810a - 243\end{aligned}$$

Check Your Understanding

1. Expand, then simplify $(3b - 1)^4$.

THINK FURTHER

In *Example 1*, how can you use the expansion of $(2a - 3)^5$ to determine the expansion of $(3a - 2)^5$?

In Lesson 8.5, the first 5 rows of Pascal's triangle were written using ${}_nC_r$ notation.

${}_0C_0$	Row 1	1
${}_1C_0 \quad {}_1C_1$	Row 2	1 1
${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$	Row 3	1 2 1
${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$	Row 4	1 3 3 1
${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$	Row 5	1 4 6 4 1

In general, the coefficients of the expansion of $(x + y)^n$ can be obtained from the terms in row $(n + 1)$ of Pascal's triangle:
${}_nC_0, {}_nC_1, {}_nC_2, \ldots {}_nC_{n-1}, {}_nC_n$
The expansion of $(x + y)^n$ is known as the **binomial theorem**.
The binomial theorem can be written using combinations or algebraic expressions.

The Binomial Theorem (using combinations)

$$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \ldots + {}_nC_{n-1}xy^{n-1} + {}_nC_ny^n$$

The Binomial Theorem (using algebraic expressions)

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + y^n$$

Example 2	**Using the Binomial Theorem to Expand**

Expand and simplify $(2c^4 - 1)^3$.

SOLUTION

Expand using combinations.

$$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \ldots + {}_nC_{n-1}xy^{n-1} + {}_nC_ny^n$$

Substitute: $n = 3, x = 2c^4, y = -1$

$$\begin{aligned}(2c^4 - 1)^3 &= {}_3C_0(2c^4)^3 + {}_3C_1(2c^4)^{3-1}(-1) + {}_3C_2(2c^4)^{3-2}(-1)^2 + {}_3C_3(-1)^3\\ &= {}_3C_0(2c^4)^3 + {}_3C_1(2c^4)^2(-1) + {}_3C_2(2c^4)^1(-1)^2 + {}_3C_3(-1)^3\\ &= 1(8c^{12}) + 3(4c^8)(-1) + 3(2c^4)(1) + 1(-1)\\ &= 8c^{12} - 12c^8 + 6c^4 - 1\end{aligned}$$

Check Your Understanding

2. Expand and simplify $(4a^2 + 2b)^3$.

The binomial theorem can be used to determine the value of a specific term in an expansion, without having to complete the expansion.

To determine the kth term in the expansion of $(x + y)^n$, use the patterns in the binomial expansion.
The kth term has:
Coefficient: ${}_nC_{k-1}$
Power of y: $k - 1$
Power of x: n minus power of y, or $n - (k - 1)$
So, an expression for the kth term is: ${}_nC_{k-1}x^{n-(k-1)}y^{k-1}$

General Term in the Expansion of $(x + y)^n$

The general term, or kth term in the expansion of $(x + y)^n$ is:
${}_nC_{k-1}x^{n-(k-1)}y^{k-1}$

Example 3 Determining a Specific Term in an Expansion

Determine the 7th term in the expansion of $(x - 3)^9$.

SOLUTION

The kth term is: ${}_nC_{k-1}x^{n-(k-1)}y^{k-1}$
Substitute: $k = 7$, $n = 9$, $y = -3$

$$t_7 = {}_9C_{7-1}x^{9-(7-1)}(-3)^{7-1}$$
$$= {}_9C_6x^3(-3)^6$$
$$= 84x^3(729)$$
$$= 61\ 236x^3$$

The 7th term is $61\ 236x^3$.

Check Your Understanding

3. Determine the 9th term in the expansion of $(x - 2)^{10}$.

Discuss the Ideas

1. What is the relationship between the coefficients of the terms in the expansion of $(x + y)^n$ and the terms in a row of Pascal's triangle? Use an example to explain.

2. How can you determine a specific term in the expansion of the power of a binomial?

Exercises

A

3. Expand using Pascal's triangle.

a) $(x + 1)^5$

b) $(x - 1)^6$

c) $(x + y)^4$

d) $(x - y)^8$

4. Determine each missing number in the expansion of $(x + y)^7$.
$x^7 + \square x^6y + 21x^5y^2 + 35x^{\square}y^3 + \square x^3y^4 + 21x^{\square}y^{\square} + 7xy^6 + y^{\square}$

5. Determine the indicated term in each expansion.

a) the last term in $(x + 1)^9$

b) the 1st term in $(x - 1)^{12}$

B

6. **a)** Multiply 4 factors of $(x - 5)$.

b) Use the binomial theorem to expand $(x - 5)^4$.

c) Compare the two methods. What conclusions can you make?

7. Expand using the binomial theorem.

a) $(x + 2)^6$

b) $(x^2 - 3)^5$

c) $(3x - 2)^4$

d) $(-2 + 2x)^4$

e) $(-4 + 3x^4)^5$

8. a) Write the terms in row 7 of Pascal's triangle.

b) Use your answer to part a to write the first 3 terms in each expansion.

i) $(x - 3)^6$

ii) $(a + 4b)^6$

iii) $(-2a + 1)^6$

iv) $(2x + 5y^2)^6$

9. Determine the coefficient of each term.

a) x^5 in $(x + 1)^8$

b) x^9 in $(x + y)^9$

c) x^2y in $(x + y)^3$

d) x^2y^3 in $(x + y)^5$

10. Explain why the coefficients of the 3rd term and the 3rd-last term in the expansion of $(x + y)^n$, $n \geq 2$, are the same.

11. Determine the indicated term in each expansion.

a) the last term in $(3x + 2)^5$

b) the 1st term in $(-2x + 5)^7$

c) the 2nd term in $(3x - 3)^4$

d) the 6th term in $(4x + 1)^8$

12. When will the coefficients of the terms in the expansion of $(ax + b)^n$ be the same as the terms in row $(n + 1)$ of Pascal's triangle?

13. Expand and simplify $(x + 1)^8 + (x - 1)^8$. What strategy did you use?

14. **a)** Show that the expansion of $(-2x + 1)^6$ is the same as the expansion of $(2x - 1)^6$.

b) Will $(-ax + b)^n$ always have the same expansion as $(ax - b)^n$? Explain.

15. Which binomial power when expanded results in $16x^4 - 32x^3 + 24x^2 - 8x + 1$?
What strategy did you use to find out?

16. Expand using the binomial theorem.

a) $(0.2x - 1.2y)^5$

b) $\left(\frac{3}{8}a + \frac{1}{6}b\right)^4$

C

17. Determine the 3rd term in the expansion of $(x^2 + 2x + 1)^6$.

18. **a)** Show that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_{n-1} + {}_nC_n = 2^n$

b) What does the relationship in part a indicate about the sum of the terms in any row of Pascal's triangle?

Multiple-Choice Questions

1. Which is the correct expansion of $(2x + 3)^3$?

A. $8x^3 + 36x^2 + 54x + 27$ **B.** $8x^3 + 54x^2 + 36x + 27$

C. $8x^3 + 63x^2 + 45x + 27$ **D.** $6x^3 + 36x^2 + 54x + 9$

2. Which statement is false about the expansion of $(x - 3)^5$?

A. The last term is -243. **B.** The 1st term is x^5.

C. The 2nd term is $15x^4$. **D.** The 3rd term is $90x^3$.

3. Which is the 7th term of $(-x - 1)^9$?

A. $-9x^8$ **B.** $-36x^7$ **C.** $-84x^3$ **D.** $-126x^5$

Study Note

What are the advantages of using the binomial theorem to expand a binomial? What are some disadvantages?

ANSWERS

Check Your Understanding

1. $81b^4 - 108b^3 + 54b^2 - 12b + 1$ **2.** $64a^6 + 96a^4b + 48a^2b^2 + 8b^3$ **3.** $11\,520x^2$

Exercises

3. a) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ **b)** $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$
c) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
d) $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$
4. 7, 4, 35, 2, 5, 7 **5. a)** 1 **b)** x^{12} **6. a), b)** $x^4 - 20x^3 + 150x^2 - 500x + 625$
7. a) $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
b) $x^{10} - 15x^8 + 90x^6 - 270x^4 + 405x^2 - 243$
c) $81x^4 - 216x^3 + 216x^2 - 96x + 16$ **d)** $16 - 64x + 96x^2 - 64x^3 + 16x^4$
e) $-1024 + 3840x^4 - 5760x^8 + 4320x^{12} - 1620x^{16} + 243x^{20}$
8. a) 1, 6, 15, 20, 15, 6, 1 **b) i)** $x^6 - 18x^5 + 135x^4$ **ii)** $a^6 + 24a^5b + 240a^4b^2$
iii) $64a^6 - 192a^5 + 240a^4$ **iv)** $64x^6 + 960x^5y^2 + 6000x^4y^4$ **9. a)** 56 **b)** 1 **c)** 3
d) 10 **11. a)** 32 **b)** $-128x^7$ **c)** $-324x^3$ **d)** $3584x^3$
13. $2x^8 + 56x^6 + 140x^4 + 56x^2 + 2$ **15.** $(-2x + 1)^4$ or $(2x - 1)^4$
16. a) $0.000\,32x^5 - 0.0096x^4y + 0.1152x^3y^2 - 0.6912x^2y^3 + 2.0736xy^4 - 2.488\,32y^5$
b) $\frac{81}{4096}a^4 + \frac{9}{256}a^3b + \frac{3}{128}a^2b^2 + \frac{1}{144}ab^3 + \frac{1}{1296}b^4$ **17.** $66x^{10}$

Multiple Choice

1. A **2.** C **3.** C

Additional Workspace

STUDY GUIDE

Concept Summary

Big Ideas	Applying the Big Ideas
• Counting strategies can be used to determine the number of ways to choose objects from a set or to arrange a set of objects.	This means that: • When the number of objects is small, graphic organizers such as organized lists and tree diagrams can be used to itemize and count arrangements. • The fundamental counting principle can be used to determine the number of ways objects can be chosen or arranged.
• A permutation is an arrangement of a set of objects where order matters. A combination is a selection from a set of objects where order does not matter.	• Different rules involving factorial notation can be used to determine the number of permutations depending on whether all the objects are different or some are identical. • The number of permutations of r objects chosen from n objects is related to the number of combinations.
• Combinations can be used to expand a power of a binomial and to generate the terms in Pascal's triangle.	• The terms in one row of Pascal's triangle can be used to determine the terms in the next row. • The coefficients of the terms in the expansion of $(x + y)^n$ correspond to the terms in row $(n + 1)$ of Pascal's triangle. • The terms in each row of Pascal's triangle and the coefficients of the terms in the expansion of $(x + y)^n$ are the same when read from left to right or from right to left because ${}_nC_r = {}_nC_{n-r}$.

Chapter Study Notes

- How do you decide whether a situation involves a permutation or a combination? How do you decide which expression or formula to use?

- What strategies can you use to expand the power of a binomial?

Skills Summary

Skill	Description	Example
Determine the number of ways to arrange n objects with or without repetition. **(8.2)**	When all objects are different, use $n!$, where n is the number of objects. When some objects are identical, use the fundamental counting principle.	The number of ways that all the letters in the word FUN can be arranged is: $3! = 3 \cdot 2 \cdot 1$ $= 6$
Determine the number of permutations of n distinct objects taken r at a time. **(8.2)**	When order matters, use the rule ${}_nP_r = \frac{n!}{(n-r)!}$, $n \geq r$, where n and r are whole numbers.	The number of 3-digit permutations of the digits in 12 579 is: ${}_5P_3 = \frac{5!}{(5-3)!}$ $= \frac{5!}{2!}$ $= 5 \cdot 4 \cdot 3$ $= 60$
Determine the number of permutations of objects of multiple kinds. **(8.3)**	When order matters, use the expression $\frac{n!}{n_1!n_2!n_3! \ldots n_k!}$, where n is the number of objects in a set with: n_1 objects of one kind; n_2 objects of another kind; n_3 objects of another kind; and so on, for k kinds of objects.	The number of 6-letter permutations of COFFEE is: $\frac{6!}{2!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{2!}$ $= \frac{360}{2}$ $= 180$
Determine the number of combinations of n distinct objects taken r at a time. **(8.4)**	When order does not matter, use the rule ${}_nC_r = \frac{n!}{(n-r)!r!}$, $n \geq r$, where r is the number of objects chosen from n distinct objects.	The number of sets of 3 books that can be chosen from 5 books is: ${}_5C_3 = \frac{5!}{(5-3)!3!}$ $= \frac{5 \cdot 4}{2!}$ $= 10$
Determine the nth term of the expansion of the binomial $(x + y)^n$. **(8.6)**	Use the general term ${}_nC_{k-1}x^{n-(k-1)}y^{k-1}$, where n is the exponent of the binomial power and k is the term number.	The 6th term in the expansion of $(x - 1)^8$ is: ${}_8C_{6-1}x^{8-(6-1)}(-1)^{6-1}$ $= {}_8C_5x^3(-1)^5$ $= -56x^3$

REVIEW

8.1

1. A penny, a dime, and a loonie are in one bag. A nickel, a quarter, and a toonie are in another bag. Tessa removes 1 coin from each bag. Use a graphic organizer to list the total amounts she could have removed.

2. The Braille code consists of patterns of raised dots arranged in a 3 by 2 array. The pattern for the letter Z is shown. How many different patterns are possible? Explain your reasoning.

Z

8.2

3. A hiking group consists of 12 students and 2 leaders. A leader must be at the front and back of the line. How many ways can the group hike in a line?

4. A code consists of 4 letters from the English alphabet and 3 letters from the Greek alphabet. There are 8 English letters and 6 Greek letters to choose from and repetition is not allowed. How many 7-letter codes are possible?

8.3

5. How many 12-letter permutations of GOBBLEDEGOOK can be created?

6. How many ways are there to arrange all the words in this tongue twister?
CAN YOU CAN A CAN AS A CANNER CAN CAN A CAN?

7. How many 9-letter permutations of EQUATIONS can be created if the vowels must appear together in the order A, E, I, O, and U? Explain.

8.4

8. A student volunteers at a food bank. He fills hampers with these items:
5 cans of soup chosen from 7 different soups
3 bags of pasta chosen from 4 different types of pasta
4 bags of vegetables chosen from 8 different types of vegetables
3 boxes of cereal chosen from 6 different types of cereal
How many ways can the student fill a hamper?

9. Solve each equation for n or r.

a) ${}_nC_3 = 84$ **b)** ${}_7C_r = 35$

8.5

10. a) These are the first 7 numbers in row 14 of Pascal's triangle:
1, 13, 78, 286, 715, 1287, 1716
Complete the row. What strategy did you use?

b) Use your results from part a to write the numbers in row 15 of the triangle.

11. Determine the value of each number in Pascal's triangle.

a) the 4th number in row 13

b) the 5th number in row 21

8.6

12. The 4th term in the expansion of $(x + 1)^8$ is $56x^5$. Which other term in the expansion has a coefficient of 56? Explain.

13. Expand using the binomial theorem.

a) $(4x - 1)^5$

b) $(5x^2 + 2y)^4$

14. Determine the indicated term in each expansion.

a) the 3rd term in $(-7x + 1)^6$

b) the 7th term in $(6x + 2y)^7$

15. Use the binomial theorem to evaluate $(1.2)^6$. Use a calculator to verify your result.

ANSWERS

1. 6¢, 26¢, \$2.01, 15¢, 35¢, \$2.10, \$1.05, \$1.25, \$3.00 **2.** 64 **3.** 958 003 200
4. 201 600 **5.** 9 979 200 **6.** 110 880 **7.** 120 **8.** 117 600 **9. a)** $n = 9$
b) $r = 3$ or $r = 4$ **10. a)** 1716, 1287, 715, 286, 78, 13, 1
b) 1, 14, 91, 364, 1001, 2002, 3003, 3432, 3003, 2002, 1001, 364, 91, 14, 1
11. a) 220 **b)** 4845 **12.** The 6th term
13. a) $1024x^5 - 1280x^4 + 640x^3 - 160x^2 + 20x - 1$
b) $625x^8 + 1000x^6y + 600x^4y^2 + 160x^2y^3 + 16y^4$ **14. a)** $36\,015x^4$ **b)** $2688xy^6$
15. 2.985 984

PRACTICE TEST

1. **Multiple Choice** A bed and breakfast has 6 rooms and 4 guests. No guests share a room. How many ways can the guests be assigned to rooms?

   A. 4! **B.** ${}_6P_4$ **C.** ${}_6P_2$ **D.** ${}_6C_4$

2. **Multiple Choice** What is the 3rd term in the expansion of $(2x - 2)^7$?

   A. $896x$ **B.** $-2688x^2$ **C.** $2688x^5$ **D.** $-896x^6$

3. A battery has a negative and a positive end. In how many different ways can 4 AAA batteries be arranged end to end? Explain.

4. **a)** Would you use a permutation or combination to solve this problem? Explain.
   In a particular week, there are 2 volleyball games, 3 floor hockey games, and 4 basketball games scheduled in Jerome's school. He has a ticket that allows him to attend 3 of the games. How many ways can Jerome attend exactly 2 floor hockey games and one other game?

   b) Solve the problem.

5. How many different ways are there to arrange all the letters in the word NANNURALUK, an Inuit word for polar bear?

6. A golfer has 13 clubs in her bag. She practises with 4 clubs from the bag. How many choices of 4 clubs can the golfer make?

7. Solve each equation.

a) ${}_nP_2 = 110$ **b)** ${}_nC_3 = 364$

8. These are the terms in row 5 of Pascal's triangle.

1 4 6 4 1

a) What are the terms in row 6?

b) Use the terms in row 6 to expand the binomial $(x - 1)^5$.

ANSWERS

1. B **2.** C **3.** 16 **4. a)** combination **b)** 18 **5.** 151 200 **6.** 715 **7. a)** $n = 11$ **b)** $n = 14$ **8. a)** 1, 5, 10, 10, 5, 1 **b)** $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$