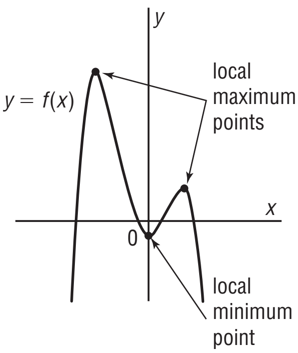
- A point where the graph changes from decreasing to increasing is called a **local minimum point**. The y-value of this point is less than those of neighbouring points.



An inspection of the graphs of polynomial functions in this lesson and in Lesson 1.3 illustrates that the graph of a polynomial function of degree n can have at most n x-intercepts and at most $(n - 1)$ local maximum or minimum points. For example, the graph of a cubic function can have at most 3 x-intercepts and at most 2 local maximum or local minimum points.

To sketch the graph of a polynomial function, use a table of values and knowledge of the end behaviour of its graph.

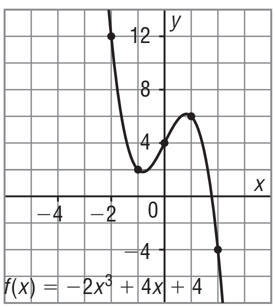
Example 1 Using a Table of Values to Sketch the Graph of a Polynomial Function

Sketch the graph of each polynomial function.

a) $f(x) = -2x^3 + 4x + 4$ **b)** $g(x) = x^4 - x^3 - 3x^2$

SOLUTION

a) The equation represents an odd-degree polynomial function. Since the leading coefficient is negative, as $x \to -\infty$, the graph rises and as $x \to \infty$, the graph falls. The constant term is 4, so the y-intercept is 4. Use a table of values to create the graph.



x	$f(x)$
−2	12
−1	2
0	4
1	6
2	−4

b) The equation represents an even-degree polynomial function. Since the leading coefficient is positive, the graph opens up. The constant term is 0, so the y-intercept is 0. Use a table of values to create the graph.

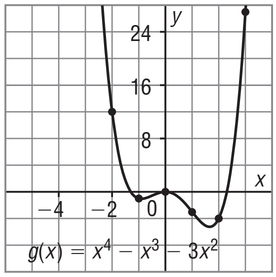
Remember that a sketch does not have to be accurate.

Check Your Understanding

1. Sketch the graph of each polynomial function.

a) $f(x) = x^3 - 3x^2$

b) $g(x) = -x^4 - 6x^3 - 9x^2 + 3$



x	$g(x)$
−2	12
−1	−1
0	0
1	−3
2	−4
3	27

Example 2 Using Intercepts to Sketch the Graph of a Polynomial Function

Check Your Understanding

2. Sketch the graph of the polynomial function:
$f(x) = x^3 + x^2 - 6x$

Sketch the graph of the polynomial function:
$f(x) = 2x^4 - x^3 - 14x^2 + 19x - 6$

SOLUTION

Factor the polynomial. Use the factor theorem.
List the factors of the constant term, -6:
$1, -1, 2, -2, 3, -3, 6, -6$
Use mental math. When $x = 1$, $f(1) = 0$
So, $x - 1$ is a factor of $2x^4 - x^3 - 14x^2 + 19x - 6$.
Divide to determine the other factor.

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -14 & 19 & -6 \\ & & 2 & 1 & -13 & 6 \\ \hline & 2 & 1 & -13 & 6 & 0 \end{array}$$

So, $2x^4 - x^3 - 14x^2 + 19x - 6 = (x - 1)(2x^3 + x^2 - 13x + 6)$
Factor the cubic polynomial. Use the factor theorem.
Let $P(x) = 2x^3 + x^2 - 13x + 6$
Use mental math.
When $x = 1$, $P(1) \neq 0$
When $x = -1$, $P(-1) \neq 0$
So, neither $x - 1$ nor $x + 1$ is a factor.
Try $x = 2$: $P(2) = 2(2)^3 + (2)^2 - 13(2) + 6$
$= 16 + 4 - 26 + 6$
$= 0$
So, $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$.
Divide to determine the other factor.

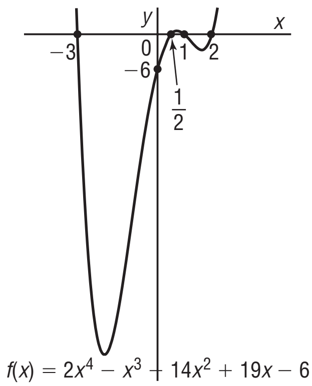
$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$2x^4 - x^3 - 14x^2 + 19x - 6 = (x - 1)(x - 2)(2x^2 + 5x - 3)$

Factor the trinomial: $2x^2 + 5x - 3 = (x + 3)(2x - 1)$
So, $f(x) = (x - 1)(x - 2)(x + 3)(2x - 1)$
Determine the zeros of $f(x)$. Let $f(x) = 0$.
$0 = (x - 1)(x - 2)(x + 3)(2x - 1)$ Solve the equation.

So, $x - 1 = 0$ or $x - 2 = 0$ or $x + 3 = 0$ or $2x - 1 = 0$
$x = 1$ $x = 2$ $x = -3$ $x = \frac{1}{2}$

The zeros are: $1, 2, -3, \frac{1}{2}$
So, the x-intercepts of the graph are: $1, 2, -3$, and $\frac{1}{2}$
Plot points at the intercepts.
The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is positive, so the graph opens up.
The constant term is -6, so the y-intercept is -6.
Draw a smooth curve through the points, beginning at the top left and ending at the top right.



THINK FURTHER

Suppose the graph of a polynomial function is symmetrical about the y-axis. What do you know about the function?

In *Example 2*, the equation $0 = (x - 1)(x - 2)(x + 3)(2x - 1)$ is the factored form of a **polynomial equation**. This *Example* illustrates that when the equation of a polynomial function is factorable, the x-intercepts of its graph can be determined by factoring.
The x-intercepts are the zeros of the polynomial function because they are the values of x when the function is 0. The zeros of the function are the roots of the related polynomial equation.

A polynomial equation may have a repeated root. Here are two examples:

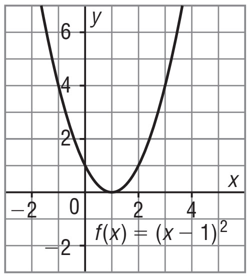
$x^2 - 2x + 1 = 0$ can be written as $(x - 1)^2 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 2, so 1 is a root with **multiplicity** 2.
The related function has a zero of multiplicity 2.

$x^3 - 3x^2 + 3x - 1 = 0$ can be written as $(x - 1)^3 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 3, so 1 is a root with multiplicity 3.
The related function has a zero of multiplicity 3.

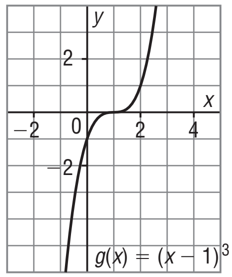
The behaviour of the graph at a zero depends on its multiplicity.

Here are the graphs of $f(x) = (x - 1)^2$ and $g(x) = (x - 1)^3$.

This graph has a zero of multiplicity 2.



This graph has a zero of multiplicity 3.



Both graphs have x-intercept 1.
The graph of $f(x) = (x - 1)^2$ touches the x-axis at $x = 1$, but does not cross the axis at this point.
The graph of $g(x) = (x - 1)^3$ crosses the x-axis at $x = 1$.
This difference in behaviour is related to the multiplicity of the zero.
In general, when a zero has even multiplicity, the graph touches the x-axis at the related x-intercept, but does not cross it; we say that the graph "just touches" the x-axis.
When a zero has odd multiplicity, the graph crosses the x-axis at the related x-intercept.

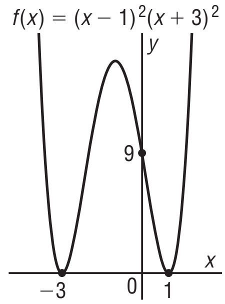
Example 3 Using the Multiplicity of a Zero to Graph a Polynomial Function

Sketch the graph of each polynomial function.

a) $f(x) = (x - 1)^2(x + 3)^2$ **b)** $g(x) = -(x + 2)^3(x - 1)^2$

SOLUTION

a) $f(x) = (x - 1)^2(x + 3)^2$



To determine the zeros, solve $f(x) = 0$.

$0 = (x - 1)^2(x + 3)^2$

The roots of the equation are $x = 1$ and $x = -3$.

So, the zeros of the function are 1 and -3.

The zero 1 has multiplicity 2.

The zero -3 has multiplicity 2.

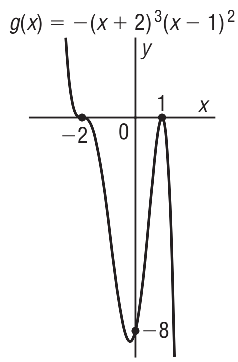
So, the graph just touches the x-axis at $x = 1$ and at $x = -3$.

The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is positive, so the graph opens up.

The y-intercept is: $(-1)^2(3)^2 = 9$

Plot points at the intercepts, then draw a smooth curve that rises to the left and rises to the right.

b) $g(x) = -(x + 2)^3(x - 1)^2$



To determine the zeros, solve $g(x) = 0$.

$0 = -(x + 2)^3(x - 1)^2$

The roots of the equation are $x = -2$ and $x = 1$.

So, the zeros of the function are -2 and 1.

The zero -2 has multiplicity 3.

The zero 1 has multiplicity 2.

So, the graph crosses the x-axis at $x = -2$, and just touches the x-axis at $x = 1$.

The equation has degree 5, so it is an odd-degree polynomial function.

The leading coefficient is negative, so as $x \to -\infty$, the graph rises and as $x \to \infty$, the graph falls.

The y-intercept is: $-(2)^3(-1)^2 = -8$

Plot points at the intercepts, then draw a smooth curve that rises to the left and falls to the right.

Check Your Understanding

3. Sketch the graph of each polynomial function.

a) $f(x) = (x + 1)^4(x - 2)$

b) $g(x) = -(x + 1)^3(x - 3)$

Discuss the Ideas

1. Suppose $x - a$ is a factor of a polynomial. What else do you know about the corresponding polynomial equation and the graph of the corresponding polynomial function?

2. How does the multiplicity of a zero of a polynomial function affect its graph?

Exercises

A

3. Which functions are polynomial functions? Justify your choices.

a) $f(x) = 2\sqrt{x} - x^2$

b) $g(x) = 6x^3 - x^2 + 3x - 7$

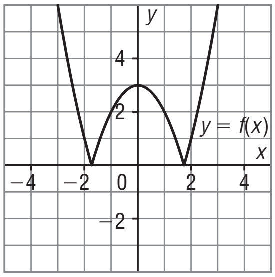
c) $h(x) = 7x^2 + 2x^3 - x - \frac{1}{2}$

d) $k(x) = 3^x + 5$

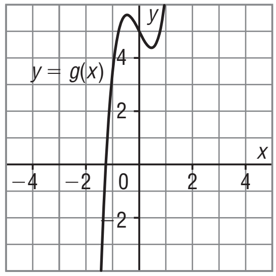
e) $p(x) = 5x^2 - 7x + \frac{2}{x}$

4. Which graphs are graphs of polynomial functions? Justify your answers.

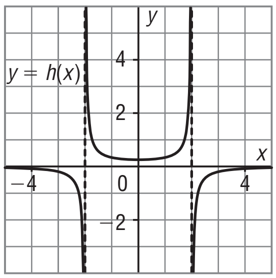
a)



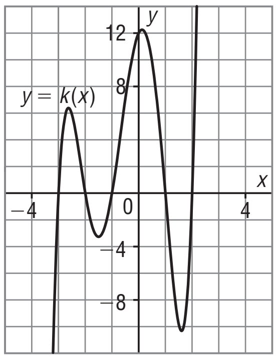
b)



c)



d)



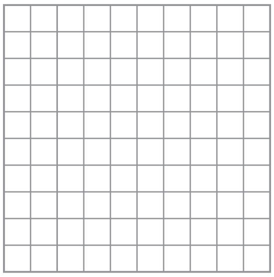
5. Complete the table below. The first row has been done for you.

	Equation	Degree	Odd or Even Degree	Type	Leading coefficient	*y*-intercept of its graph
	$f(x) = 3x^2 - 2x + 1$	2	even	quadratic	3	1
a)	$g(x) = 5x + x^5 - 2x^3$					
b)	$h(x) = 2x^2 - 3x^3 - 7$					
c)	$k(x) = 5 - x^4 - 3x$					

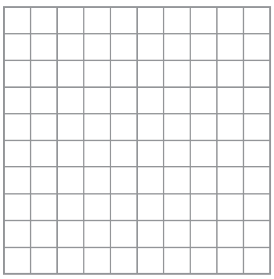
B

6. Use a table of values to sketch the graph of each polynomial function.

a) $f(x) = x^3 - 7x + 6$



b) $g(x) = -x^4 + 5x^2 - 4$



7. Use intercepts to sketch the graph of each polynomial function.

a) $f(x) = 2x^3 + 3x^2 - 2x$

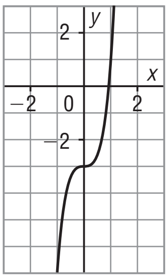
b) $h(x) = 2x^4 + 7x^3 + 4x^2 - 7x - 6$

8. Identify the graph that corresponds to each function. Justify your choices.

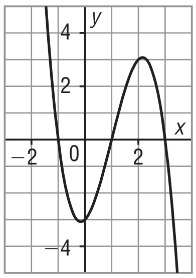
a) $f(x) = -x^3 + 3x^2 + x - 3$ **b)** $g(x) = x^4 - 3x^2 - 3$

c) $h(x) = x^5 + 3x^3 - 3$ **d)** $k(x) = -x^2 + 4x - 3$

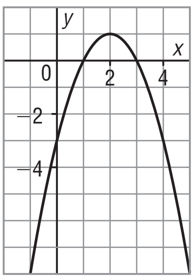
i) Graph A



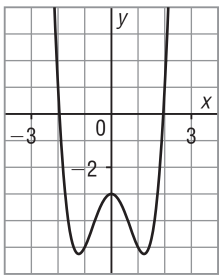
ii) Graph B



iii) Graph C



iv) Graph D



9. Determine the zeros of each polynomial function. State the multiplicity of each zero. How does the graph of each function behave at the related x-intercepts?
Use graphing technology to check.

a) $f(x) = (x + 3)^3$ **b)** $g(x) = (x - 2)^2(x + 3)^2$

c) $h(x) = (x - 1)^4(2x + 1)$ **d)** $j(x) = (x - 4)^3(x + 1)^2$

10. Sketch the graph of this polynomial function.
$h(x) = (x + 1)^2(x - 1)(x + 2)$

11. a) Write an equation in standard form for each polynomial function described below.

i) a cubic function with zeros 3, -3, and 0

ii) a quartic function with zeros -2 and 1 of multiplicity 1, and a zero 2 of multiplicity 2

b) Is there more than one possible equation for each function in part a? Explain.

12. Sketch a possible graph of each polynomial function.

a) cubic function; leading coefficient is positive; zero of 4 has multiplicity 3

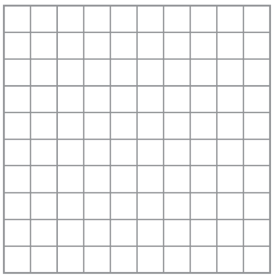
b) quintic function; leading coefficient is positive; zero of 3 has multiplicity 2; zero of -2 has multiplicity 2; zero of -4 has multiplicity 1

c) quartic function; leading coefficient is negative; zero of -4 has multiplicity 3; zero of 3 has multiplicity 1

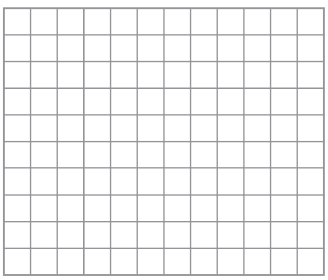
13. A cubic function has zeros 2, 3, and -1. The y-intercept of its graph is -18. Sketch the graph, then determine an equation of the function.

C

14. Investigate pairs of graphs of even-degree polynomial functions of the form shown below for different values of the variables a, b, c, and $d \in \mathbb{Z}$. Describe one graph as a transformation image of the other graph. What conclusions can you make?
$h(x) = (x + a)(x + b)(x + c)(x + d)$ and
$k(x) = (x - a)(x - b)(x - c)(x - d)$



15. Investigate pairs of graphs of odd-degree polynomial functions of the form shown below for different values of the variables a, b, c, d, and $e \in \mathbb{Z}$. Describe one graph as a transformation image of the other graph. What conclusions can you make?
$h(x) = (x + a)(x + b)(x + c)(x + d)(x + e)$ and
$k(x) = (x - a)(x - b)(x - c)(x - d)(x - e)$



16. Each of the functions $f(x) = x^3 - 27x + 54$ and $g(x) = x^3 - 27x - 54$ has one zero of multiplicity 2 and one different zero. Use only this information to determine the values of b for which the function $h(x) = x^3 - 27x + b$ has each number of zeros. Explain your strategy.

a) 3 different zeros

b) 1 zero of multiplicity 1 and no other zeros

Multiple-Choice Questions

1. The graph of a polynomial function rises to the left and falls to the right. Which statement describes the function?

A. The function has an odd degree and its leading coefficient is positive.

B. The function has an even degree and its leading coefficient is positive.

C. The function has an odd degree and its leading coefficient is negative.

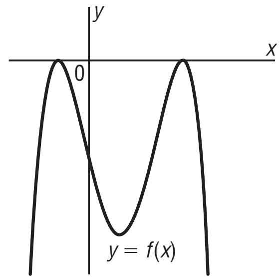
D. The function has an even degree and its leading coefficient is negative.

2. The graph of a polynomial function of degree 4 is shown. Which statements are true?

I. The function has an even degree.
II. The function has a zero of multiplicity 3.
III. The y-intercept is negative.
IV. The function is a quartic function.

A. I, II, IV **B.** I, III, IV

C. I, II **D.** I, III



Study Note

Generalize the rules for graphing polynomial functions of odd or even degree.

ANSWERS

Exercises

3. a) no **b)** yes **c)** yes **d)** no **e)** no **4. a)** no **b)** yes **c)** no **d)** yes
5. a) 5; odd; quintic; 1; 0 **b)** 3; odd; cubic; −3; −7 **c)** 4; even; quartic; −1; 5
8. a) B **b)** D **c)** A **d)** C **9. a)** −3, multiplicity 3
b) 2, multiplicity 2; −3, multiplicity 2 **c)** 1, multiplicity 4; −0.5, multiplicity 1
d) 4, multiplicity 3; −1, multiplicity 2 **11. b)** yes
13. $y = -3x^3 + 12x^2 - 3x - 18$ **16. a)** $-54 < b < 54$ **b)** $b > 54$ or $b < -54$

Multiple Choice

1. C **2.** B

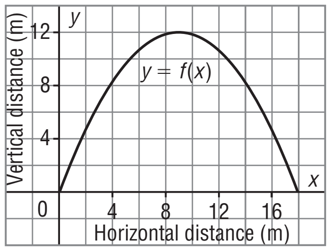
Additional Workspace

1.5 Modelling and Solving Problems with Polynomial Functions

FOCUS **Analyze the graph of a polynomial function to solve a problem.**

Get Started

This graph represents the path of a ball kicked into the air.



What was the maximum height of the ball?

How far had the ball travelled horizontally when it reached this height?

What total horizontal distance did the ball travel?

Construct Understanding

Use graphing technology.
A certain airline's regulations state that the sum of the length, width, and depth of a piece of carry-on luggage must not exceed 100 cm. Several models of carry-on luggage have length 10 cm greater than their depth.
Write a polynomial function to represent the volume of this luggage. Graph the function.
To the nearest cubic centimetre, what is the maximum possible volume of this luggage? What are its dimensions to the nearest tenth of a centimetre? What assumptions did you make?

THINK FURTHER

In the problem above, suppose the sum of the length, width, and depth was halved. Would the maximum volume also be halved? Explain.

Polynomial functions and their graphs can be used to solve real-world problems.

Example 1 Using the Graph of a Cubic Function to Solve a Problem

A piece of sheet metal 25 cm long and 18 cm wide is used to make a box with no lid. Equal squares of side length x centimetres are cut from the corners and the sides are folded up.

a) Write a polynomial function to represent the volume, V, of the box in terms of x.
b) Graph the function. What is the domain?
c) To the nearest cubic centimetre, what is the maximum volume of the box? What size of square should be cut out to create a box with this volume? To the nearest tenth of a centimetre, what are the dimensions of the box?

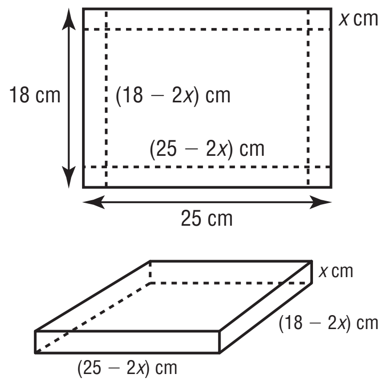
Check Your Understanding

1. A piece of cardboard 30 cm long and 25 cm wide is used to make a box with no lid. Equal squares of side length x centimetres are cut from the corners and the sides are folded up.

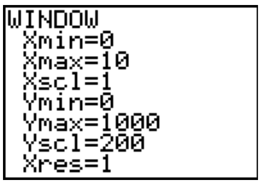
a) Write a polynomial function to represent the volume, V, of the box in terms of x.

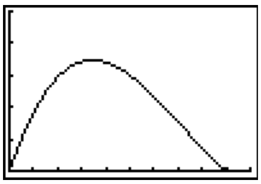
SOLUTION

a) Sketch a diagram.
The box has height x centimetres, width $(18 - 2x)$ centimetres, and length $(25 - 2x)$ centimetres.
So, a polynomial function that represents the volume, V, of the box is: $V(x) = x(18 - 2x)(25 - 2x)$

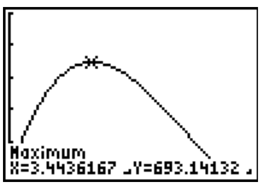


b) Use a graphing calculator. Enter the equation:
$y = x(18 - 2x)(25 - 2x)$
The dimensions of the box are positive.
The sheet metal has width 18 cm.
So, the side length of a square cut from each corner must be less than $\frac{18 \text{ cm}}{2}$, or 9 cm.
So, the domain is: $0 < x < 9$
Use these window settings:





c) To determine the maximum volume, press: $\boxed{\text{2nd}}\boxed{\text{TRACE}}\boxed{4}$, then use the arrow keys to move the cursor to determine the approximate coordinates of the local maximum point.



So, the maximum volume of the box is approximately 693 cm^3.
This occurs when each square that is cut out has a side length of approximately 3.4 cm.
The approximate dimensions of the box are:
Height: 3.4 cm
Width: 18 cm − 2(3.4 cm) = 11.2 cm
Length: 25 cm − 2(3.4 cm) = 18.2 cm
So, for maximum volume, the dimensions of the box are approximately 18.2 cm by 11.2 cm by 3.4 cm.

b) Graph the function. What is the domain?

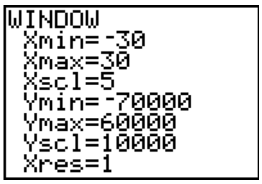
c) To the nearest cubic centimetre, what is the maximum volume of the box? What size of square should be cut out to create a box with this volume? To the nearest tenth of a centimetre, what are the dimensions of the box?

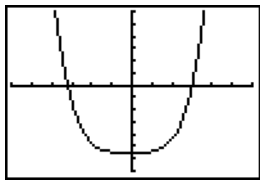
Example 2 Using the Graph of a Quartic Function to Solve a Problem

Leo and 3 friends each have birthdays on December 13. Sanda is 3 years younger than Leo. Leo is 4 years younger than Vince. Hunter is 1 year older than Leo. On December 13, 2010, the product of their ages was 54 658 greater than the sum of their ages. How old was Leo and each friend on that day?

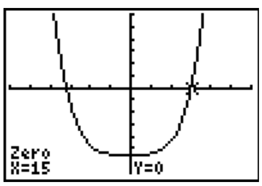
SOLUTION

Let Leo's age in years be x.
Then, Sanda's age in years is $x - 3$, Vince's age in years is $x + 4$, and Hunter's age in years is $x + 1$. The sum of their ages is:
$x + (x - 3) + (x + 4) + (x + 1) = 4x + 2$
Sum of ages + 54 658 = product of ages
So, $4x + 2 + 54\,658 = x(x - 3)(x + 4)(x + 1)$
$0 = x(x - 3)(x + 4)(x + 1) - 4x - 54\,660$
Enter the equation $y = x(x - 3)(x + 4)(x + 1) - 4x - 54\,660$ into a graphing calculator. Graph the function using these window settings:





Since age cannot be negative, the positive x-intercept of the graph represents Leo's age.
To determine the positive x-intercept, press: 2nd TRACE 2, then use the arrow keys. The x-intercept is 15.
So, Leo was 15 years old on that day.



The ages of the other friends, in years, were: Sanda: $15 - 3 = 12$
Vince: $15 + 4 = 19$
Hunter: $15 + 1 = 16$
On December 13, 2010, Leo was 15, Sanda was 12, Vince was 19, and Hunter was 16.

Check Your Underanding

2. Clara and 3 friends were born on March 11. Lesley is 5 years younger than Clara. Mike is 2 years younger than Clara. Thomas is 3 years older than Clara. On March 11, 2011, the product of their ages was 61 136 greater than the sum of their ages. How old was Clara and each friend on that day?

Discuss the Ideas

1. Why might the domain and range of a polynomial function differ from the domain and range of a function that models a situation using the same polynomial?

2. When you use a graphing calculator to graph a polynomial function, how do you determine the window settings?

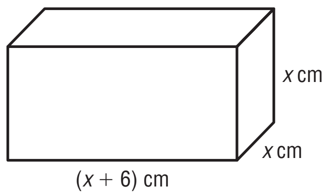
Exercises

Use technology to graph the functions.

A

3. A box has length $(x + 2)$ units, width $(x - 5)$ units, and height $(2x + 3)$ units. Write a polynomial function to represent its volume, V, in terms of x.

4. **a)** Write a polynomial function to represent the volume, V, of this square prism in terms of x.



b) Graph the function. Sketch the graph. Use the graph to determine the dimensions of the prism if its volume is 81 cm^3.

B

5. A piece of cardboard 36 cm long and 28 cm wide is used to make an open box. Equal squares of side length x centimetres are cut from the corners and the sides are folded up.

a) Write expressions to represent the length, width, and height of the box in terms of x.

b) Write a polynomial function to represent the volume of the box in terms of x.

c) Graph the function. Sketch the graph. What is the domain?

d) What is the maximum volume of the box? What is the side length of the square that should be cut out to create a box with this volume? Give your answers to the nearest tenth.

6. The volume, in cubic centimetres, of an expandable gift box can be represented by the polynomial function $V(x) = -x^3 + 35x^2 + 200x$. The height of the box in centimetres is $40 - x$. Assume the length is greater than the width.

a) Determine binomial expressions for the length and width of the box in terms of x.

b) Graph the function. Sketch the graph. What do the x-intercepts represent?

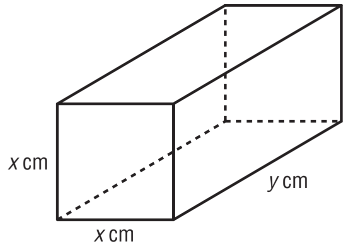
c) To the nearest cubic centimetre, what is the maximum volume of the gift box?

7. Fred and Ted are twins. They were born 3 years after their older sister, Bethany. This year, the product of their three ages is 5726 greater than the sum of their ages. How old are the twins?

8. Ann, Stan, and Fran are triplets. They were born 4 years before their sister, Kim. This year, the product of their four ages is 49 092 greater than the sum of their ages. How old is Kim?

9. A carton of juice has dimensions 6.4 cm by 3.8 cm by 10.9 cm. The manufacturer wants to design a box with double the capacity by increasing each dimension by x centimetres. To the nearest tenth of a centimetre, what are the dimensions of the larger carton?

10. A package sent by a courier has the shape of a square prism. The sum of the length of the prism and the perimeter of its base is 100 cm.



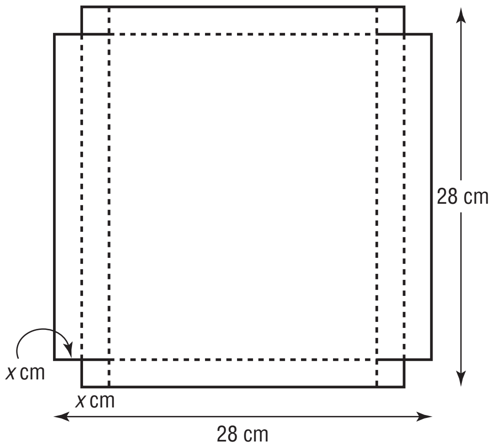
a) Write a polynomial function to represent the volume V of the package in terms of x.

b) Graph the function. Sketch the graph.

c) To the nearest tenth of a centimetre, what are the dimensions of the package for which its volume is maximized?

C

11. An open box with locking tabs is to be made from a square piece of cardboard with side length 28 cm. This is done by cutting equal squares of side length x centimetres from the corners and folding along the dotted lines as shown.



a) Write a polynomial function to represent the volume, V, of the box in terms of x.

b) Graph the function. Sketch the graph. State the domain.

c) To the nearest centimetre, what is the value of x for the box with maximum volume?

12. A manufacturer designs a cylindrical can with no top. The surface area of the can is 300 cm^2. The can has base radius r centimetres.

a) Write a polynomial function to model the capacity, C cubic centimetres, of the can as a function of r.

b) Graph the function. Sketch the graph. To the nearest tenth of a centimetre, what are the radius and height of the can when it has a maximum capacity?

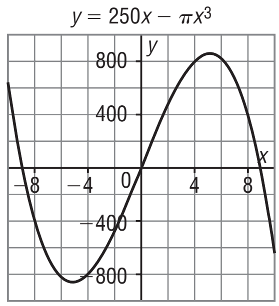
Multiple-Choice Questions

1. A box has the shape of a rectangular prism. The height of the box is x centimetres. The length of the box is $4x$ centimetres. The sum of its length, width, and height is 18 cm. What is a polynomial function that represents the volume, V, of the box?

A. $V(x) = x(4x)(18 - 5x)$ **B.** $V(x) = x(4x)(18 - 4x)$

C. $V(x) = x(4x)(18 - x)$ **D.** $V(x) = x(18 - 4x)(18 - 5x)$

2. Here is the graph of $y = 250x - \pi x^3$. Suppose y is the volume of a closed cylindrical can with radius x centimetres and surface area 500 cm^2. Which statements are correct?



I. The maximum volume is approximately 858 cm^3.

II. The minimum volume is approximately -858 cm^3.

III. The volume is 0 when x is approximately 9 cm.

IV. The can has no maximum volume.

A. I and II **B.** I and III

C. I and IV **D.** II and IV

Study Note

Which characteristics of the graph of a polynomial function may be used to solve a problem, involving volume, that is modelled by a polynomial function?

ANSWERS

Check Your Understanding

1. a) $V(x) = x(25 - 2x)(30 - 2x)$ **b)** $0 < x < 12.5$ **c)** 1512 cm^3; 4.5 cm; 21.0 cm by 16.0 cm by 4.5 cm **2.** In years: Clara: 17; Lesley: 12; Mike: 15; Thomas: 20

Exercises

3. $V(x) = (x + 2)(x - 5)(2x + 3)$ **4. a)** $V(x) = x^2(x + 6)$
b) 3 cm by 3 cm by 9 cm **5. a)** height: x centimetres; length: $(36 - 2x)$ centimetres; width: $(28 - 2x)$ centimetres **b)** $V(x) = x(36 - 2x)(28 - 2x)$ **c)** $0 < x < 14$
d) approximately 2342.9 cm^3; approximately 5.2 cm
6. a) length: $(x + 5)$ centimetres; width: x centimetres
c) approximately 11 284 cm^3 **7.** 17 years old **8.** 12 years old
9. 8.0 cm by 5.4 cm by 12.5 cm **10. a)** $V(x) = 4x^2(25 - x)$
c) approximately 16.7 cm by 16.7 cm by 33.3 cm **11. a)** $V(x) = 8x(7 - x)(14 - x)$
b) $0 < x < 7$ **c)** 3 cm **12. a)** $C(r) = \frac{300r - \pi r^3}{2}$ **b)** radius = height $\doteq$ 5.6 cm

Multiple Choice

1. A **2.** B

Additional Workspace

STUDY GUIDE

Concept Summary

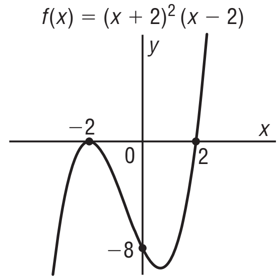
Big Ideas	Applying the Big Ideas
• Some polynomials can be factored by using long division.	This means that: • Polynomials can be divided by other polynomials. • When a polynomial P(x) is divided by a binomial of the form $x - a$, $a \in \mathbb{Z}$, the remainder is P(a). • A binomial is a factor of a polynomial when the division results in a remainder of 0.
• The zeros of a polynomial function or the x-intercepts of its graph can be determined by solving the corresponding polynomial equation.	• When a polynomial can be factored, the zeros of the related polynomial function or the x-intercepts of its graph can be determined by equating each factor to 0. • The behaviour of the graph of a polynomial function at its x-intercepts depends on the multiplicity of each zero of the function: – When a zero has even multiplicity, the graph just touches the x-axis at the related x-intercept. – When a zero has odd multiplicity, the graph crosses the x-axis at the related x-intercept.

Chapter Study Notes

• What do you need to remember when you divide a polynomial by a binomial?

• What characteristics of the graph of a polynomial function can be determined from its equation?

Skills Summary

Skill	Description	Example
Divide a polynomial by a binomial of the form $x - a$, $a \in \mathbb{Z}$. **(1.1)**	To divide using synthetic division, remove the variables and record only the coefficients. Include a 0 for any term that is not included in the polynomial.	Divide $4x^3 - 13x + 9$ by $x + 2$. $\begin{array}{r\|rrrr} -2 & 4 & 0 & -13 & 9 \\ & & -8 & 16 & -6 \\ \hline & 4 & -8 & 3 & 3 \end{array}$ So, $(4x^3 - 13x + 9) \div (x + 2)$ $= 4x^2 - 8x + 3$ R3
Use the remainder and factor theorems, and the factor property to factor a polynomial. **(1.2)**	If $x - a$ is a factor of a polynomial, then a is a factor of the constant term in the polynomial. The binomial $x - a$ is a factor of $P(x)$ if $P(a) = 0$.	Determine a binomial factor of $P(x) = 8x^3 + 6x^2 - 11x - 3$ The factors of -3 are: 1, -1, 3, -3 Try $x = 1$: $P(1) = 8(1)^3 + 6(1)^2 - 11(1) - 3$ $= 0$ So, $x - 1$ is a factor.
Sketch the graph of a polynomial function. **(1.4)**	In the equation of a polynomial function, the leading coefficient determines the end behaviour of the graph and the constant term is the y-intercept of the graph. The zeros of the function are the x-intercepts of the graph. The multiplicity of a zero determines the behaviour of the graph at the related x-intercept.	Sketch a graph of $f(x) = (x + 2)^2(x - 2)$. x-intercepts: -2, 2 The zero -2 has multiplicity 2; the graph just touches the x-axis at $x = -2$. The zero 2 has multiplicity 1; the graph crosses the x-axis at $x = 2$. The equation has degree 3, so it is an odd-degree polynomial function. The leading coefficient is positive, so as $x \to -\infty$, the graph falls and as $x \to \infty$, the graph rises. The constant term is -8, so the y-intercept is -8. $f(x) = (x + 2)^2 (x - 2)$ 
Solve a problem by analyzing the graph of a polynomial function. **(1.5)**	Write a polynomial function to model the situation, then graph the function and analyze the graph to solve the problem.	A rectangular prism has width x metres, length $(x + 2)$ metres, and height $(x + 5)$ metres. What are the dimensions of a prism with volume 120 m^3? A polynomial function that models the volume is: $y = x(x + 2)(x + 5)$ Graph the function and $y = 120$. Determine the x-coordinate of the point of intersection: (3, 120) The prism has dimensions 3 m by 5 m by 8 m.

REVIEW

1.1

1. Use long division to divide $7x^3 + 6x^4 - 7x - 9x^2 + 8$ by $x - 1$. Write the division statement.

2. Use synthetic division to divide. Write the division statement.

a) $(2x^2 - 2x + x^3 - 3x^4 + 5) \div (x - 1)$

b) $(-x^4 + 4x^5 - 16 - 4x) \div (1 + x)$

1.2

3. Determine the remainder when $x^4 - x^3 - 11x^2 + 9x + 18$ is divided by each binomial. Which binomials are factors of the polynomial? How do you know?

a) $x + 2$

b) $x + 3$

4. For each polynomial, determine one factor of the form $x - a$, $a \in \mathbb{Z}$.

a) $4x^3 - 5x^2 - 23x + 6$

b) $9x^4 - 37x^2 + 4$

5. Factor: $x^3 - 5x^2 - 2x + 24$

1.3

6. a) For each polynomial function below, predict the end behaviour of the graph. Justify your prediction, then use graphing technology to check.

i) $g(x) = -2x^3 + 5x^2 - 8$

ii) $h(x) = -x^4 + 2x^3 - 5x^2 + 9$

iii) $k(x) = x^4 - 2x^3 + 5x^2 - 9$

b) What do you notice about the graphs and equations of the functions in part a, ii and iii?

7. Match each function to its graph. Justify your choices.

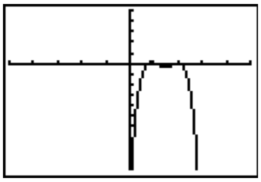
a) $y = x^4 - x^2 - 25x - 12$

b) $y = -x^4 + 8x^3 - 23x^2 + 28x - 12$

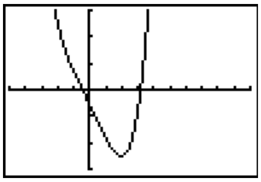
c) $y = x^5 - 3x^2 + 5$

d) $y = -x^3 + 2x + 5$

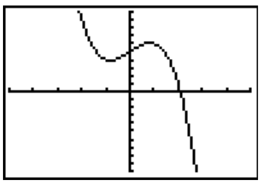
i) Graph A



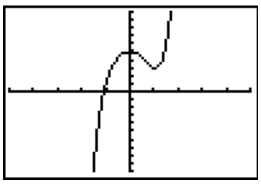
ii) Graph B



iii) Graph C



iv) Graph D



1.4

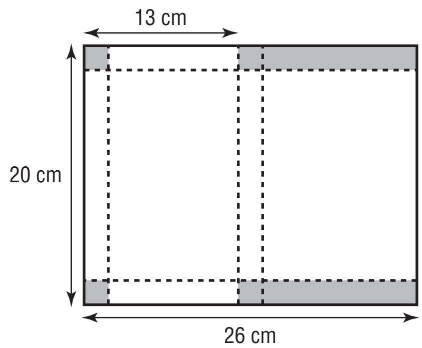
8. Sketch the graph of each polynomial function.

a) $g(x) = -x^4 - 3x^3 + 11x^2 + 3x - 10$

b) $f(x) = (x - 3)(x - 1)(x + 2)^2$

1.5

9. A piece of cardboard 26 cm long and 20 cm wide is used to make a gift box that has a top. The diagram shows the net for the box. The shaded parts are discarded. The squares cut from each corner have side length x centimetres. What is the maximum volume of the box? What is the side length of the square that should be cut out to create a box with this volume? Give your answers to the nearest tenth.



ANSWERS

1. $6x^4 + 7x^3 - 9x^2 - 7x + 8 = (x - 1)(6x^3 + 13x^2 + 4x - 3) + 5$

2. a) $-3x^4 + x^3 + 2x^2 - 2x + 5 = (x - 1)(-3x^3 - 2x^2 - 2) + 3$

b) $4x^5 - x^4 - 4x - 16 = (x + 1)(4x^4 - 5x^3 + 5x^2 - 5x + 1) - 17$

3. a) -20 **b)** 0 **4. a)** $x + 2$ or $x - 3$ **b)** $x - 2$ or $x + 2$

5. $x^3 - 5x^2 - 2x + 24 = (x - 4)(x - 3)(x + 2)$ **7. a)** B **b)** A **c)** D **d)** C

9. approximately 433.6 cm^3; approximately 3.7 cm

PRACTICE TEST

1. **Multiple Choice** Which statement is true?

A. When $2x^3 + 4x^2 - 2x - 1$ is divided by $x - 2$, the remainder is 3.

B. The binomial $x + 1$ is a factor of $4x^4 - x^3 - 3x + 2$.

C. When $2x^4 - 7x^3 + 6x^2 - 14x + 20$ is divided by $x - 3$, the remainder is -5.

D. The binomial $x + 2$ is a factor of $5x^3 + 7x^2 + 12$.

2. **Multiple Choice** Which statement about the graph of a quartic function is false?

A. The graph may open up.

B. The graph may have a zero of multiplicity 3.

C. The graph may fall to the left and rise to the right.

D. The graph may have a zero of multiplicity 2.

3. Divide $2x^4 + 11x^3 - 10 - 5x + 14x^2$ by $x + 2$.
Write the division statement.

4. Does the polynomial $x^4 - x^3 - 14x^2 + x + 16$ have a factor of $x + 3$? How do you know?

5. Factor: $4x^4 - 20x^3 + 17x^2 + 26x - 15$

6. Sketch the graph of this polynomial function.
$g(x) = 4x^4 + 11x^3 - 7x^2 - 11x + 3$

7. Canada Post defines a small packet as one for which the sum of its length, width, and height is less than or equal to 90 cm. A company produces several different small packets, each with length 15 cm longer than its height.

a) Write a polynomial function to represent possible volumes of one of these packets in terms of its height x. Assume the sum of the dimensions is maximized.

b) Graph the function.

c) To the nearest cubic centimetre, what is the maximum possible volume of one of these packets? What are its dimensions to the nearest tenth of a centimetre?

ANSWERS

1. D **2.** C **3.** $2x^4 + 11x^3 + 14x^2 - 5x - 10 = (x + 2)(2x^3 + 7x^2 - 5)$
4. no **5.** $(x + 1)(x - 3)(2x - 5)(2x - 1)$ **7. a)** $V(x) = x(x + 15)(75 - 2x)$
c) 25 347 cm^3; 38.1 cm by 28.8 cm by 23.1 cm

2 Radical and Rational Functions

BUILDING ON

- graphing polynomial functions
- solving radical and rational equations algebraically
- simplifying rational expressions

BIG IDEAS

- The graph of a function, $y = f(x)$, can be used to graph the corresponding radical function, $y = \sqrt{f(x)}$.
- For the graph of a rational function, $y = \frac{f(x)}{g(x)}$, the non-permissible values of x correspond to vertical asymptotes or holes.
- The roots of a rational equation or radical equation are the x-intercepts of the graph of a corresponding function.

LEADING TO

- curve sketching using calculus

NEW VOCABULARY

radical function	rational function
invariant point	oblique (slant) asymptote

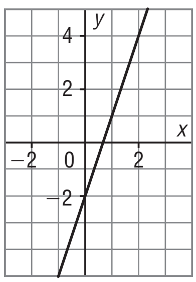
2.1 Properties of Radical Functions

FOCUS **Graph and analyze radical functions, and solve radical equations.**

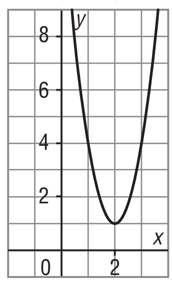
Get Started

What are the domain and range of each function?

$y = 3x - 2$



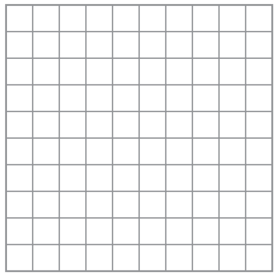
$y = 3(x - 2)^2 + 1$



Construct Understanding

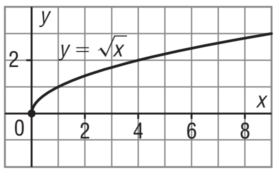
Complete the table of values for $y = x$ and $y = \sqrt{x}$.
Sketch a graph of each function, then state its domain and range.
When is the graph of $y = \sqrt{x}$ above the graph of $y = x$?
When is it below?
Suppose you are given the graph of $y = x$.
How could you use it to graph $y = \sqrt{x}$?

x	−4	−3	−2	−1	−0.25	0	0.25	1	2	3	4
$y = x$											
$y = \sqrt{x}$											



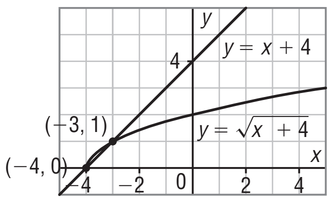
A **radical function** has the form $y = \sqrt{f(x)}$, where $f(x)$ is a function. The square root of a number is only defined for non-negative numbers, so the domain of $y = \sqrt{f(x)}$ is the set of values of x for which $f(x) \geq 0$.

Here is the graph of $y = \sqrt{x}$.
The domain of $y = \sqrt{x}$ is $x \geq 0$ and the range is $y \geq 0$.
Since both $x \geq 0$ and $y \geq 0$, the graph starts at the origin and extends in only one direction.



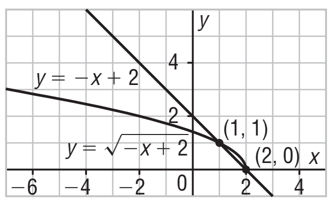
Here are the graphs of two linear functions and their related radical functions.

$y = x + 4$ and $y = \sqrt{x + 4}$



The points $(-4, 0)$ and $(-3, 1)$ lie on both graphs. Between these points, the graph of $y = \sqrt{x + 4}$ lies above the graph of $y = x + 4$.
Since $x + 4 \geq 0$
then $\quad x \geq -4$
So, the domain of $y = \sqrt{x + 4}$ is $x \geq -4$; and the range is $y \geq 0$.
The graph of $y = \sqrt{x + 4}$ opens to the right.

$y = -x + 2$ and $y = \sqrt{-x + 2}$



The points $(1, 1)$ and $(2, 0)$ lie on both graphs. Between these points, the graph of $y = \sqrt{-x + 2}$ lies above the graph of $y = -x + 2$.
Since $-x + 2 \geq 0$
then $\quad x \leq 2$
So, the domain of $y = \sqrt{-x + 2}$ is $x \leq 2$; and the range is $y \geq 0$.
The graph of $y = -\sqrt{x + 2}$ opens to the left.

The graphs at the bottom of page 83 illustrate characteristics that are common to all graphs of $y = f(x)$ and $y = \sqrt{f(x)}$.

- If 0 and 1 are in the range of $y = f(x)$, then points with these y-coordinates lie on both graphs; these are **invariant points**.
- Where the graph of $y = f(x)$ lies between the graph of $y = 1$ and the x-axis, the graph of $y = \sqrt{f(x)}$ lies above the graph of $y = f(x)$.
- Where the graph of $y = f(x)$ lies above the graph of $y = 1$, the graph of $y = \sqrt{f(x)}$ lies below the graph of $y = f(x)$.
- Where the graph of $y = f(x)$ lies below the x-axis, the graph of $y = \sqrt{f(x)}$ does not exist.

These characteristics can be used to sketch a graph of $y = \sqrt{f(x)}$ when the graph of $y = f(x)$ is given.

THINK FURTHER

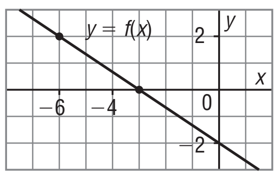
Why does the graph of $y = \sqrt{f(x)}$ lie above the graph of $y = f(x)$ between the invariant points?

Example 1 — Sketching the Graph of $y = \sqrt{f(x)}$ Given the Graph of a Linear Function $y = f(x)$

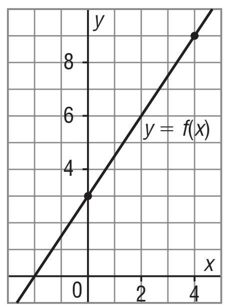
For each graph of $y = f(x)$ below:

- Sketch the graph of $y = \sqrt{f(x)}$.
- State the domain and range of $y = \sqrt{f(x)}$.

a)

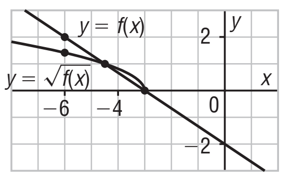


b)



SOLUTION

a) From the graph:
$f(x) \geq 0$ for all values of $x \leq -3$,
so the domain of $y = \sqrt{f(x)}$ is $x \leq -3$.
Mark the points where $y = 0$ or $y = 1$;
the graph of $y = \sqrt{f(x)}$ lies above the graph of $y = f(x)$ between these points. Identify the coordinates of another point that lies above the x-axis on the graph of $y = \sqrt{f(x)}$:



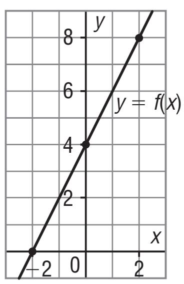
x	$y = f(x)$	$y = \sqrt{f(x)}$
−6	2	$\sqrt{2} \doteq 1.4$

Join the points with a smooth curve for the graph of $y = \sqrt{f(x)}$.
The range of $y = \sqrt{f(x)}$ is $y \geq 0$.

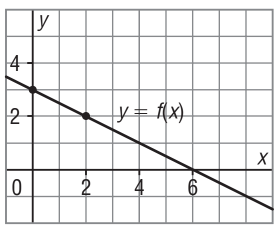
Check Your Understanding

1. For each graph of $y = f(x)$ below:
   - Sketch the graph of $y = \sqrt{f(x)}$.
   - State the domain and range of $y = \sqrt{f(x)}$.

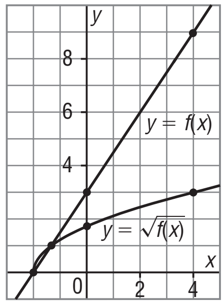
a)



b)



b) From the graph:
$f(x) \geq 0$ for all values of $x \geq -2$, so the domain of $y = \sqrt{f(x)}$ is $x \geq -2$.
Mark the points where $y = 0$ or $y = 1$. Identify the coordinates of other points that lie above the x-axis on the graph of $y = \sqrt{f(x)}$:



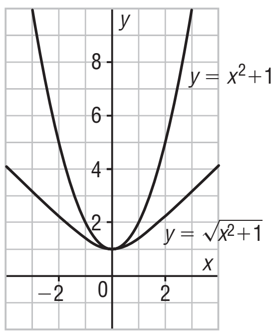
x	$y = f(x)$	$y = \sqrt{f(x)}$
0	3	$\sqrt{3} \doteq 1.7$
4	9	$\sqrt{9} = 3$

Join the points with a smooth curve for the graph of $y = \sqrt{f(x)}$.
The range of $y = \sqrt{f(x)}$ is $y \geq 0$.

In *Example 1b*, the functions $y = f(x)$ and $y = \sqrt{f(x)}$ have different domains because the square root of a negative number is not defined; and, if $x < -2$, then $f(x)$ is negative and $\sqrt{f(x)}$ is not defined. The ranges are different because $\sqrt{f(x)}$ is the principal square root of $f(x)$, which is always 0 or positive.

Here are tables of values for and graphs of $y = x^2 + 1$ and $y = \sqrt{x^2 + 1}$.

x	$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
-3	10	$\sqrt{10} \doteq 3.2$
-2	5	$\sqrt{5} \doteq 2.2$
-1	2	$\sqrt{2} \doteq 1.4$
0	1	$\sqrt{1} = 1$
1	2	$\sqrt{2} \doteq 1.4$
2	5	$\sqrt{5} \doteq 2.2$
3	10	$\sqrt{10} \doteq 3.2$



THINK FURTHER

Describe the graph of a linear function, $y = f(x)$, for which the graph of $y = \sqrt{f(x)}$ is a horizontal line, and $\sqrt{f(x)} > f(x)$ for all real values of x.

The domain of $y = x^2 + 1$ is $x \in \mathbb{R}$, and the range is $y \geq 1$.
The domain of $y = \sqrt{x^2 + 1}$ is $x \in \mathbb{R}$, and the range is $y \geq 1$.
The only invariant point is (0, 1).

THINK FURTHER

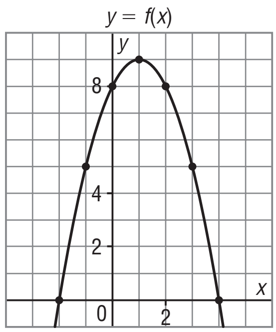
Consider the quadratic function $y = x^2 - 1$. Would the domain and range of this function be the same as the domain and range of $y = \sqrt{x^2 - 1}$?

Example 2 Sketching the Graph of $y = \sqrt{f(x)}$ Given the Graph of a Quadratic Function $y = f(x)$

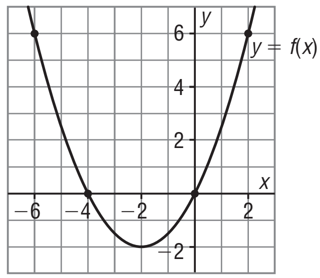
For the graph of each quadratic function $y = f(x)$ below:

- Sketch the graph of $y = \sqrt{f(x)}$.
- State the domain and range of $y = \sqrt{f(x)}$.

a)



b)

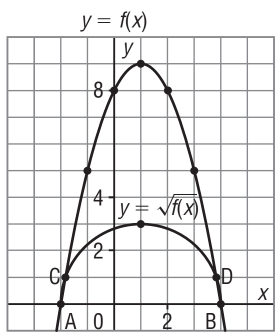


SOLUTION

a) The graph of $y = f(x)$ lies below the x-axis for $x < -2$ and for $x > 4$, so $y = \sqrt{f(x)}$ is undefined for these values of x. The domain of $y = \sqrt{f(x)}$ is $-2 \le x \le 4$. Mark points A and B where $y = 0$, and points C and D where $y = 1$. The graph of $y = \sqrt{f(x)}$ lies above the graph of $y = f(x)$ between A and C, and between B and D. Identify the coordinates of another point with x-coordinate in the domain of $y = \sqrt{f(x)}$.

x	$y = f(x)$	$y = \sqrt{f(x)}$
1	9	$\sqrt{9} = 3$

Join the points with a smooth curve for the graph of $y = \sqrt{f(x)}$.



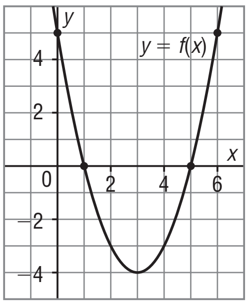
The range of $y = \sqrt{f(x)}$ is $0 \le y \le 3$.

Check Your Understanding

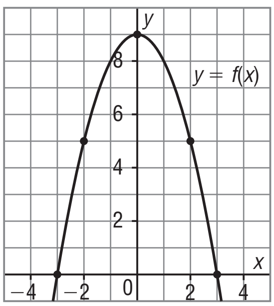
2. For the graph of each quadratic function $y = f(x)$ below:

- Sketch the graph of $y = \sqrt{f(x)}$.
- State the domain and range of $y = \sqrt{f(x)}$.

a)



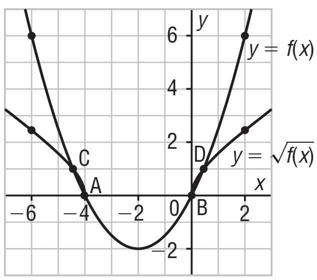
b)



b) The graph of $y = f(x)$ lies below the x-axis for $-4 < x < 0$, so $y = \sqrt{f(x)}$ is undefined for these values of x. The domain of $y = \sqrt{f(x)}$ is $x \leq -4$ or $x \geq 0$. Mark points A and B where $y = 0$ and points C and D where $y = 1$. The graph of $y = \sqrt{f(x)}$ lies above the graph of $y = f(x)$ between A and C, and between B and D. Identify the coordinates of other points with x-coordinates in the domain of $y = \sqrt{f(x)}$.

x	$y = f(x)$	$y = \sqrt{f(x)}$
−6	6	$\sqrt{6} \doteq 2.4$
2	6	$\sqrt{6} \doteq 2.4$

Join the points with two smooth curves for the graph of $y = \sqrt{f(x)}$.



The range of $y = \sqrt{f(x)}$ is $y \geq 0$.

To sketch the graph of a radical function, it may not always be necessary to determine the coordinates of additional points beyond those with y-coordinates 0 and 1.

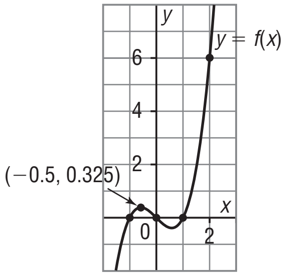
THINK FURTHER

The graph of a radical function $y = \sqrt{g(x)}$ is the point (2, 0). What is an explicit equation for a possible quadratic function $y = g(x)$?

Example 3 Sketching the Graph of $y = \sqrt{f(x)}$ Given the Graph of a Cubic Function $y = f(x)$

For the graph of the cubic function $y = f(x)$:

- Sketch the graph of $y = \sqrt{f(x)}$.
- State the domain and range of $y = \sqrt{f(x)}$.

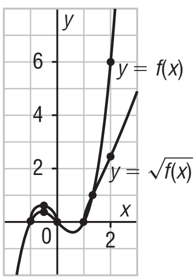


SOLUTION

The graph of $y = f(x)$ lies below the x-axis for $x < -1$ and for $0 < x < 1$, so the graph of $y = \sqrt{f(x)}$ is undefined for these values of x.
The domain of $y = \sqrt{f(x)}$ is $-1 \leq x \leq 0$ or $x \geq 1$.
Mark the three points where $y = 0$ and the point where $y = 1$.
Identify the coordinates of other points on the graph of $y = \sqrt{f(x)}$.

x	$y = f(x)$	$y = \sqrt{f(x)}$
−0.5	0.325	$\sqrt{0.325} \doteq 0.6$
2	6	$\sqrt{6} \doteq 2.4$

Join the points with two smooth curves for the graph of $y = \sqrt{f(x)}$.

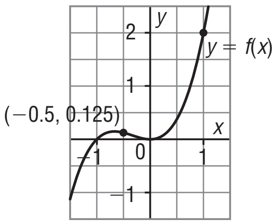


The range of $y = \sqrt{f(x)}$ is $y \geq 0$.

Check Your Understanding

3. For the graph of the cubic function $y = f(x)$:

- Sketch the graph of $y = \sqrt{f(x)}$.
- State the domain and range of $y = \sqrt{f(x)}$.



The graph of a radical function can be used to solve a related radical equation. The zeros of the graph are the roots of the equation.