first steps®

First Steps in Mathematics

Measurement

Resource Book 2

Indirect Measure
Estimate

Improving the mathematics outcomes of students

PEARSON

Government of Western Australia
Department of Education and Training

FSIM011 | First Steps in Mathematics: Measurement Book 2

First Steps in Mathematics: Measurement Resource Book 2
Indirect Measure and Estimate

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Diagnostic Map: Measurement

Emergent Phase

During the Emergent Phase

Students initially attend to overall appearance of size, recognizing one thing as perceptually bigger than another and using comparative language in a fairly undifferentiated and absolute way (big/small) rather than to describe comparative size (bigger/smaller). Over time, they note that their communities distinguish between different forms of bigness (or size) and make relative judgements of size.

As a result, they begin to understand and use the everyday language of attributes and comparison used within their home and school environment, differentiating between attributes that are obviously perceptually different.

By the end of the Emergent phase, students typically:

- distinguish tallness, heaviness, fatness, and how much things hold
- start to distinguish different forms of length and to use common contextual length distinctions; e.g., distinguish wide from tall
- use different bipolar pairs to describe things; e.g., thin—fat, heavy—light, tall—short
- describe two or three obvious measurement attributes of the same thing; e.g., tall, thin, and heavy
- describe something as having more or less of an attribute than something else, e.g., as being taller than or as being fatter than.

Matching and Comparing Phase

Most students will enter the Matching and Comparing phase between 5 and 7 years of age.

As students move from the Emergent phase, to the Matching and Comparing phase, they:

- may not "conserve" measures; e.g., thinking that moving a rod changes its length, pouring changes "how much," cutting up paper makes more surface area
- may visually compare the size of two things, but make no effort to match; e.g., saying which stick is longer without lining up the bases or which sheet of paper is bigger without superimposing
- compare time spans but may not take into account different starting times; e.g., deciding that the TV program that finished latest was on longest
- use bipolar pairs but may have difficulty with some comparative terms; e.g., lift to decide which is heavier but say both are heavy because both hands go down
- may distinguish two attributes (such as tallness and mass) but not understand that the two attributes may lead to different orders of size for a collection, expecting the order for tallness and the order for mass to be the same
- while describing different attributes of the same thing (tall, thin, and heavy) may be confused by a request to compare two things by different attributes, particularly if the comparisons lead to different orders
- often do not think to use counting to say how big or how much bigger; e.g., they may "weigh" something by putting it into one side of a balance and smaller objects into the other side but not count the objects

During the Matching and Comparing Phase

Students match in a conscious way in order to decide which is bigger by familiar readily perceived and distinguished attributes such as length, mass, capacity, and time. They also repeat copies of objects, amounts, and actions to decide how many fit (balance or match) a provided object or event.

As a result, they learn to directly compare things to decide which is longer, fatter, heavier, holds more, or took longer. They also learn what people expect them to do in response to questions such as "How long (tall, wide or heavy, much time, much does it hold)?" or when explicitly asked to measure something.

By the end of the Matching and Comparing phase, students typically:

- attempt to focus on a particular attribute to compare two objects or events; e.g., how much the jar holds
- know that several objects or events may be in different orders when compared by different attributes
- line up the base of two sticks when comparing their lengths and fit regions on top of each other to compare area
- use the everyday notion of "how many fit" and count how many repeats of an object fit into or match another; e.g., How many pens fit along the table? How many potato prints cover the sheet? How many blocks fit in the box?
- count units and call it "measuring;" e.g., *I measured and found the jar holds a bit more than 7 scoops.*
- use "between" to describe measurements of uni-dimensional quantities (length, mass, capacity, time); e.g., *It weights between 7 and 8 marbles.*
- refer informally to part-units when measuring uni-dimensional quantities; e.g., *Our room is 6 and a bit metres long.*

Quantifying Phase

Most students will enter the Quantifying phase between 7 and 9 years of age.

As students move from the Matching and Comparing phase to the Quantifying phase, they:

- while knowing that ordering objects by different attributes may lead to different orders, may still be influenced by the more dominant perceptual features; e.g., they may still think the tallest container holds the most
- may count "units" in order to compare two things but be fairly casual in their repetition of units, not noticing gaps or overlaps; e.g., placing the first "unit" away from the end when measuring length, not worrying about spills when measuring how much a container holds, not stopping their claps immediately the music stops
- do not necessarily expect the same "answer" each time when deciding how many fit
- many not think to use unit information to answer questions such as: Which cup holds more? Will the table slide through the door?
- may not see the significance of using a common unit to compare two things and, when using different units, let the resulting number override their perceptual judgement
- while many will have learned to use the centimetre marks on a conventional rule to "measure" lengths, they often do not see the connection between the process and the repetition of units

During the Quantifying Phase

Students connect the two ideas of directly comparing the size of things and of deciding "how many fit" and so come to an understanding that the count of actual or imagined repetitions of units gives an indication of size and enables two things to be compared without directly matching them.

As a result, they trust information about repetitions of units as an indicator of size and are prepared to use this in making comparisons of objects.

By the end of the Quantifying phase, students typically:

- attempt to ensure uniformity of representations of the unit; e.g., check that the cup is always full, the pencil does not change length, the balls are the same size
- use the representations of their unit carefully to make as close a match as possible, avoiding gaps and overlaps; e.g., choose a flexible tape to measure the perimeter of a curved shape
- know why they need to choose the same size objects to use as units when comparing two quantities
- see repeating one representation of the unit over and over as equivalent to filling or matching with multiple copies of it
- connect the repetition of a 'unit' with the numbers on a whole-number calibrated scale
- make things to a specified length in uniform units (including centimetres and metres)
- use provided measurements to make a decision about comparative size; e.g., use the fact that a friend's frog weighs 7 marbles to decide whether their own frog is heavier or lighter
- count units as a strategy to solve comparison problems such as: Whose frog is heavier? Put the jars in order from the one that holds the most to the one that holds the least.
- are prepared to say which is longer (heavier) based on information about the number of units matching each object
- think of different things having the same "size"; e.g., use grid paper to draw different shapes with the same perimeter
- add measurements that they can readily think of in terms of repetitions of units; e.g., find the perimeter of a shape by measuring the sides and adding

What is the Diagnostic Map for Measurement?

How students currently think about measurement attributes and units will influence how they respond to the activities provided for them, and hence what they are able to learn from them. As students' thinking about measurement develops, it goes through a series of characteristic phases. Recognizing these common patterns of thinking should help you to interpret students' responses to activities, to understand why they seem to be able to do some things and not others, and also why some students may be having difficulty in achieving certain outcomes while others are not. It should also help you to provide the challenges students need to move their thinking forward, refine their half-formed ideas, overcome any misconceptions they might have to and hence achieve the outcomes.

See over

Diagnostic Map: Measurement cont.

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- add measurements that they can readily think of in terms of repetitions of units; e.g., find the perimeter of a shape by measuring the sides and adding

Measuring Phase

Most students will enter the Measuring phase between 9 and 11 years of age.

As students move from the Quantifying phase to the Measuring phase, they:

- while trying to make as close a match as possible to the thing to be measured, may find the desire to match closely overriding the need for consistency of unit; e.g., they may resort to "filling" a region with a variety of different objects in order to cover it as closely as possible
- may not understand that the significance of having no gaps and overlaps is that the "true" measurement is independent of the placement of the units
- may still think of the unit as an object and of measuring as "fitting" in the social sense of the word (How many people fit in the elevator? How many beans in the jar?) and so have difficulty with the idea of combining part-units as is often needed in order to find the area of a region
- many confuse the unit (a quantity) with the instrument (or object) used to represent it; e.g., they may think a square metre has to be a square with sides of 1 m, may count cubes for area and not think of the face of each as the unit
- may interpret whole numbered marks on a calibrated scale as units but may not interpret the meaning of unlabelled graduations

During the Measuring Phase

Students come to understand the unit as an amount (rather than an object or a mark on a scale) and to see the process of matching a unit with an object as equivalent to subdividing the object into bits of the same size as the unit and counting the bits.

As a result, they see that part-units can be combined to form whole units and they understand and trust the measurement as a property or description of the object being measured that does not change as a result of the choice or placement of units.

By the end of the Measuring phase, students typically:

- expect the same number of copies of the representation of their unit to match the object being measured regardless of how they arrange or place the copies
- understand that the smaller the unit the greater the number; e.g., are able to say which is the longer of a 1-km walk and a 1400-m walk.
- compose "part-units" into wholes, understanding, for example, that a narrow garden bed may have an area of 5 or 6 m^2 even though no whole "metre squares" fit into the bed
- can themselves partition a rectangle into appropriate squares and use the array structure to work out how many squares are in the rectangle
- interpret the unnumbered graduations on a familiar whole-number scale
- understand the relationship between "part-units" and the common metric prefixes; e.g., know that a unit can be broken into one hundred parts and each part will be a centi-unit
- work with provided measurement information alone; e.g., order measurements of capacity provided in different standard units, make things that meet measurement specifications

Relating Phase

Most students will enter the Relating phase between 11 and 13 years of age

As students move from the Measuring phase to the Relating phase, they:

- while partitioning a rectangle into appropriate squares and using the array structure to find its area, may not connect this with multiplying the lengths of the sides of a rectangle to find its area
- while understanding the inverse relationship between the unit and the number of units needed, may still be distracted by the numbers in measurements and ignore the units; e.g., say that 350 g is more than 2 kg
- while converting between known standard units, may treat related metric measures just as they would any other unit, not seeing the significance of the decimal structure built into all metric measures

During the Relating Phase

Students come to trust measurement information even when it is about things they cannot see or handle and to understand measurement relationships, both those between attributes and those between units.

As a result, they work with measurement information itself and can use measurements to compare things, including those they have not directly experienced, and to indirectly measure things.

By the end of the Relating phase, students typically:

- understand that known relationships between attributes can be used to find measurements that cannot be found directly; e.g., understand that we can use length measurements to work out area
- know that for figures of the same shape (that is, similar) the greater the length measures the greater the area measures, but this is not so if the figures are different shapes
- understand why the area of a rectangle and the volume of a rectangular prism can be found by multiplying its length dimensions and can use this for fractional side lengths
- think of the part-units themselves as units; e.g., a particular unit can be divided into one hundred parts and each part is then a centi-unit
- subdivide units to make measurements more accurate
- choose units that are sufficiently small (that is, accurate) to make the needed comparisons
- use their understanding of the multiplicative structure built into the metric system to move flexibly between related standard units; e.g., they interpret the 0.2 kg mark on a scale as 200 g
- notice and reject unrealistic estimates and measurements, including of objects or events they have not actually seen or experienced
- use relationships between measurements to find measures indirectly; e.g., knowing that 1 mL = 1 cm^3 they can find the volume of an irregular solid in cubic centimetres by finding how many millilitres of water it displaces using a capacity cylinder

CONTENTS

INTRODUCTION

The *First Steps in Mathematics* resource books and professional learning program are designed to help teachers plan, implement, and evaluate the mathematics program they provide for students. The series describes the key mathematical ideas students need to understand in order to achieve the principal learning goals of mathematics curricula across Canada and around the world.

Unlike many resources that present mathematical concepts that have been logically ordered and prioritized by mathematicians or educators, *First Steps in Mathematics* follows a sequence derived from the mathematical development of real children. Each resource book is based on five years of research by a team of teachers from the Western Australia Department of Education and Training, and tertiary consultants led by Professor Sue Willis at Murdoch University.

The *First Steps in Mathematics* project team conducted an extensive review of international research literature, which revealed gaps in the field of knowledge about students' learning in mathematics. Many of these findings are detailed in the Background Notes that supplement the Key Understandings described in the *First Steps in Mathematics* resource books for Measurement.

Using tasks designed to replicate those in the research literature, team members interviewed hundreds of elementary school children in diverse locations. Analysis of the data obtained from these interviews identified characteristic phases in the development of students' thinking about mathematical concepts.

The Diagnostic Maps—which appear in the resource books for Number, Measurement, Geometry and Space, and Data Management and Probability—describe these phases of development, exposing specific markers where students often lose, or never develop, the connection between mathematics and meaning. Thus, *First Steps in Mathematics* helps teachers systematically observe not only what mathematics individual children do, but how the children do the mathematics, and how to advance the children's learning.

It has never been more important to teach mathematics well. Globalization and the increasing use of technology have created changing demands for the application of mathematics in all aspects of our lives. Teaching mathematics well to all students requires a high level of understanding of teaching and learning in mathematics and of mathematics itself. The *First Steps in Mathematics* series and professional learning program help teachers provide meaningful learning experiences and enhance their capacity to decide how best to help all students achieve the learning goals of mathematics.

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Chapter 1

An Overview of *First Steps in Mathematics*

First Steps in Mathematics is a professional learning program and series of teacher resource books that are organized around mathematics curricula for Number, Measurement, Geometry and Space, and Data Management and Probability.

The aim of *First Steps in Mathematics* is to improve students' learning of mathematics.

First Steps in Mathematics examines mathematics within a developmental framework to deepen teachers' understanding of teaching and learning mathematics. The developmental framework outlines the characteristic phases of thinking that students move through as they learn key mathematical concepts. As teachers internalize this framework, they make more intuitive and informed decisions around instruction and assessment to advance student learning.

First Steps in Mathematics helps teachers to:

- build or extend their own knowledge of the mathematics underpinning the curriculum
- understand how students learn mathematics so they can make sound professional decisions
- plan learning experiences that are likely to develop the mathematics outcomes for all students
- recognize opportunities for incidental teaching during conversations and routines that occur in the classroom

This chapter details the beliefs about effective teaching and learning that *First Steps in Mathematics* is based on and shows how the elements of the teacher resource books facilitate planning and instruction.

Beliefs about Teaching and Learning

Focus Improves by Explicitly Clarifying Outcomes for Mathematics

Learning is improved if the whole-school community has a shared understanding of the mathematics curriculum goals, and an implementation plan and commitment to achieving them. A common understanding of these long-term aims helps individuals and groups of teachers decide how best to support and nurture students' learning, and how to tell when this has happened.

All Students Can Learn Mathematics to the Best of Their Ability

A commitment to common goals signals a belief that **all** students can be successful learners of mathematics. A situation where less is expected of and achieved by certain groups of students is not acceptable. School systems, schools and teachers are all responsible for ensuring that **each** student has access to the learning conditions he or she requires to achieve the curricular goals to the best of his or her ability.

Learning Mathematics Is an Active and Productive Process

Learning is not simply about the transfer of knowledge from one person to another. Rather, students need to construct their own mathematical knowledge in their own way and at a pace that enables them to make sense of the mathematical situations and ideas they encounter. A developmental learning approach is based on this notion of learning. It recognizes that not all students learn in the same way, through the same processes, or at the same rate.

Common Curricular Goals Do Not Imply Common Instruction

The explicit statement of the curricular goals expected for all students helps teachers to make decisions about the classroom program. However, the list of content and process goals for mathematics is not a curriculum. If all students are to succeed to the best of their ability on commonly agreed concepts, different curriculum implementations will not only be possible, but also be necessary. Teachers must decide what type of instructional activities are needed for their students to achieve the learning goals.

> *A curriculum that enables all students to learn must allow for different starting points and pathways to learning so that students are not left out or behind.*
>
> ***—Darling-Hammond, National standards and assessments, p. 480***

Professional Decision-Making Is Central in Teaching

It is the responsibility of teachers to provide all students with the conditions necessary for them to achieve the curricular goals of mathematics. This responsibility requires teachers to make many professional decisions simultaneously, such as what to teach, to whom, and how, and making these professional decisions requires a synthesis of knowledge, experience, and evidence.

> *Professionalism has one essential feature; …(it) requires the exercise of complex, high level professional judgments…(which) involve various mixes of specialised knowledge; high level cognitive skills; sensitive and sophisticated personal skills; broad and relevant background and tacit knowledge.*
>
> **—Preston, Teacher professionalism, p. 2, 20**

The personal nature of each student's learning journey means that the decisions teachers make are often "non-routine," and the reasoning processes involved can be complex. These processes cannot be reduced to a set of instructions about what to do in any given situation. Teachers must have the freedom and encouragement to adapt existing curricula flexibly to best meet their students' needs and to move them forward. The improvement of students' learning is most likely to take place when teachers have good information about tasks, response range, and intervention techniques on which to base their professional decisions.

"Risk" Relates to Future Mathematics Learning

Risk cannot always be linked directly to students' current achievement. Rather, it refers to the likelihood that their future mathematical progress is "at risk."

Some students who can answer questions correctly might not have the depth of understanding needed for ongoing progress. Others might have misconceptions that could also put their future learning "at risk." A number of students may make errors that are common when they try to make sense of new mathematical ideas and, therefore, show progress. For example, a student who tries to count tiles using an array may count the corners twice, which is incorrect. However, the use of the array signals progress because the student is using his or her knowledge of the repeating nature of the area unit.

Students who are learning slowly, or whose previous experiences are atypical, might nevertheless progress steadily if their stage of learning is accommodated with appropriate, but challenging, learning experiences.

Successful Mathematics Learning Is Robust Learning

Robust learning, which focuses on students developing mathematics concepts fully and deeply, is essential if learning is to be sustained over the long term.

A focus on short-term performance or procedural knowledge at the expense of robust knowledge places students "at risk" of not continuing to progress throughout the years of schooling.

Learning Mathematics: Implications for the Classroom

Learning mathematics is an active and productive process on the part of the learner. The following section illustrates how this approach influences the ways in which mathematics is taught in the classroom.

Learning Is Built on Existing Knowledge

Learners' interpretations of mathematical experiences depend on what they already know and understand. For example, many young students may distinguish two attributes (such as tallness and heaviness) but not understand that the two attributes may lead to different orders of size for a collection, expecting the order for height and the order for mass to be the same. Other students may compare time spans but may not take into account different starting times; e.g., deciding that the television program that finished latest was on longest.

In each case, students' existing knowledge should be recognized and used as the basis for further learning. Their learning should be developed to include the complementary knowledge with the new knowledge being linked to and building on students' existing ideas.

Learning Requires That Existing Ideas Be Challenged

Learning requires that students extend or alter what they know as a result of their knowledge being challenged or stretched in some way. For example, a challenge may occur when a student predicts that the tallest container will hold the most water, then measures and finds that it does not.

Another challenge may occur when a student believes that when size increases, mass increases. Or, it may happen when the student finds that peers think about a problem in a different way. The student must find some way of dealing with the challenge or conflict provided by the new information in order to learn.

Learning Occurs when the Learner Makes Sense of the New Ideas

Teaching is important—but learning is done *by* the learner rather than *to* the learner. This means the learner acts on and makes sense of new information. Students almost always try to do this. However, in trying to make sense of their mathematical experiences, some students will draw conclusions that are not quite what their teachers expect.

Also, when students face mathematical situations that are not meaningful, or are well beyond their current experience and reach, they often conclude that the mathematics does not make sense or that they are incapable of making sense of it. This may encourage students to resort to learning strategies based on the rote imitation of procedures. The result is likely to be short-term rather than effective long-term learning. Teachers have to provide learning experiences that are meaningful and challenging, but within the reach of their students.

Learning Involves Taking Risks and Making Errors

In order to learn, students have to be willing to try a new or different way of doing things, and stretch a bit further than they think they can. At times, mistakes can be a sign of progress. For example, students may count "units" in order to compare two things but be fairly casual in their repetition of units, not noticing gaps or overlaps; e.g., placing the first "unit" away from the end when measuring length, not worrying about spills when measuring how much a container holds, or not stopping their claps immediately when the music stops. Such errors can be positive signs that students are matching and comparing as they move to understand the more precise meaning of quantity or "how much."

Errors can provide a useful source of feedback, challenging students to adjust their conceptions before trying again. Errors may also suggest that learners are prepared to work on new or difficult problems where increased error is likely. Or, learners may try improved ways of doing things that mean giving up old and safe, but limited, strategies. For example, a student who can successfully find the number of paper tiles to tile a rectangular room by laying them all out in order takes a risk when trying to multiply the lengths of the sides of the rectangle, since multiplying may result in increased mistakes in the short term.

Learners Get Better with Practice

Students should get adequate opportunities to practise mathematics, but this involves much more than the rote or routine repetition of facts and procedures. For example, the very language of "square unit" convinces many students that units of area have to be square shapes. They will need considerable experience in cutting and rearranging shapes to convince themselves that rearranging the cut shape does not change the area and that a "square unit" is an amount of area not a shape or thing.

Likewise, if students are to develop a rich understanding of measurement they will need spaced and varied opportunities to notice and reject unrealistic estimates and measurements, including things they have not actually seen or experienced. Repetitious procedures of routine questions are unlikely to provide this rich understanding. In fact, they are more likely to interfere with it.

Learning Is Helped by Clarity of Purpose for Students as well as Teachers

Learning is likely to be enhanced if students understand what kind of learning activity they should be engaged in at any particular time. This means helping students to distinguish between tasks that provide practice of an already learned procedure and tasks that are intended to develop understanding of mathematical concepts and processes. In the former case, little that is new is involved, and tasks are repetitive, so they become habitual and almost unthinking. Students should expect to be able to start almost immediately and, if they cannot, realize that they may need to know more and seek help.

With tasks that are intended to develop understanding, non-routine tasks and new ideas may be involved. Students should not expect to know what to do or to be able to get started immediately.

Students may spend a considerable amount of time on a single task, and they will often be expected to work out for themselves what to do. They should recognize that, for such activities, persistence, thoughtfulness, struggle, and reflection are expected.

Teaching Mathematics

Teachers assume considerable responsibility for creating the best possible conditions for learning. The kind of learning tasks and environment teachers provide depend on their own view of how learning is best supported. The perspective that learning is an active and productive process has two significant implications for teaching.

First, teachers cannot predict or control exactly what and when students learn. They need to plan curricula that provide students with a wider and more complex range of information and experiences than they would be expected to understand fully at any given time. For example, teachers often provide opportunities for their students to explore the number system using calculators. Similar experiences can assist students in making parallel connections to the multiplicative structure of the metric system. Before formally introducing a concept, a teacher can present situations that challenge students to use their prior understanding of number to experiment with ideas about measurement, such as, “Is 0.2 km equivalent to 2 m, 20 m, 200 m, or 2000 m?” In this way teachers can stimulate their students’ curiosity about connections within mathematics, helping students develop notions about the structure of the metric system at many different levels preceding the prescribed teaching of these connections.

This represents a significant change in curriculum planning. It is a movement away from an approach that only exposes students to content and ideas that they should be able to understand or do at a particular point in time.

Second, for students to become effective learners of mathematics, they must be engaged fully and actively. Students will want, and be able, to take on the challenge, persistent effort, and risks involved. Equal opportunities to learn mathematics means teachers will:

- provide an environment for learning that is equally supportive of all students
- offer each student appropriate mathematical challenges
- foster in all students processes that enhance learning and contribute to successful achievment of goals

Understanding the Elements of *First Steps in Mathematics*

The elements of *First Steps in Mathematics* embody the foregoing beliefs about teaching and learning and work together to address three main questions:

- What are students expected to learn?
- How does this learning develop?
- How do teachers advance this learning?

Learning Outcomes for the Measurement Strand

The Measurement strand focuses on the basic principles of measurement: the range of measures in common use and the skills needed for everyday purposes. As a result of their learning, students will develop confidence and proficiency in using direct and indirect measurement and estimating skills to describe, compare, evaluate, plan, and construct.

To achieve these outcomes, students require an understanding of the nature of the different physical attributes that can be measured and the way units are used to quantify amounts of such attributes to needed levels of accuracy. It also requires the ability and understandings needed to make informed judgements about measurements for a range of purposes and to calculate measurement indirectly using measurement relationships. Learning experiences should be provided that will enable students to understand units, directly and indirectly measure, and estimate measurements.

As a result of their learning experiences, students at all levels should be able to achieve the following outcomes.

Understand Units

Decide what needs to be measured by selecting what attributes to measure and what units to use.

Direct Measure

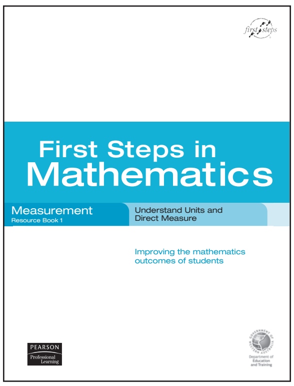
Carry out measurements of length, capacity, volume, mass, area, time, and angle to needed levels of accuracy.

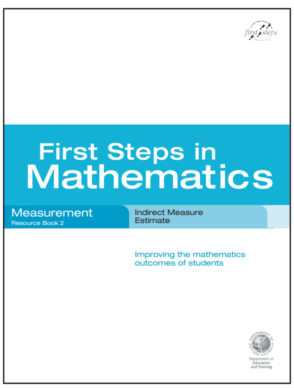
Indirect Measure

Select, interpret, and combine measurements, measurement relationships and formulas to determine other measures indirectly.

Estimate

Make sensible direct and indirect estimates of quantities and be alert to the reasonableness of measurements and results.





Integrating the Outcomes

The outcomes suggested above for Measurement are each dealt with in a separate chapter. This is to emphasize the importance of each and the difference between them. For example, students need to learn about what attributes to measure and what units to use (Understand Units) as well as developing the skill to reliably and accurately use units to directly measure each of these attributes (Direct Measure). By paying separate and special attention to each outcome, teachers can make sure that both areas receive sufficient attention and that important ideas about each are drawn out of the learning experiences they provide.

This does not mean, however, that the ideas and skills underpinning each of the outcomes should be taught separately or that they will be learned separately. The outcomes are inextricably linked. Consequently, many of the activities will provide opportunities for students to develop their ideas about more than one of the outcomes. This will help teachers to ensure that the significant mathematical ideas are drawn from the learning activities so that students achieve each of the outcomes for Measurement.

How Does This Learning Develop?

First Steps in Mathematics: Measurement describes characteristic phases in students' thinking about the major mathematical concepts of the Measurement strand. These developmental phases are organized in a Diagnostic Map.

Diagnostic Map

The Diagnostic Map for Measurement details five developmental phases. It helps teachers to:

- understand why students seem to be able to do some things and not others
- realize why some students may be experiencing difficulty while others are not
- indicate the challenges students need to move their thinking forward, to refine their preconceptions, overcome any misconceptions, and so develop deep reflective understanding about concepts
- interpret students' responses to activities

Diagnostic Map: Measurement

Emergent Phase

During the Emergent Phase

Students initially attend to overall appearance of size, recognizing one thing as perceptually bigger than another and using comparative language in a fairly undifferentiated and absolute way (big/small) rather than to describe comparative size (bigger/smaller). Over time, they note that their communities distinguish between different forms of bigness (or size) and make relative judgements of size.

As a result, they begin to understand and use the everyday language of attributes and comparison used within their home and school environment, differentiating between attributes that are obviously perceptually different.

By the end of the Emergent phase, students typically:

- distinguish tallness, heaviness, fatness, and how much

Matching and Comparing Phase

Most students will enter the Matching and Comparing phase between 5 and 7 years of age.

As students move from the Emergent phase, to the Matching and Comparing phase, they:

- may not "conserve" measures; e.g., thinking that moving a rod changes its length, pouring changes "how much," cutting up paper makes more surface area
- may visually compare the size of two things, but make no effort to match; e.g., saying which stick is longer without lining up the bases or which sheet of paper is bigger without superimposing
- compare time spans but may not take into account different starting times; e.g., deciding that the TV program that finished latest was on longest
- use bipolar pairs but may have difficulty with some comparative terms; e.g., lift to decide which is heavier but say both are heavy because both hands go down
- may distinguish two attributes (such as tallness and mass) but not understand that the two attribut

Quantifying Phase

Most students will enter the Quantifying phase between 7 and 9 years of age.

As students move from the Matching and Comparing phase to the Quantifying phase, they:

- while knowing that ordering objects by different attributes may lead to different orders, may still be influenced by the more dominant perceptual features; e.g., they may still think the tallest container holds the most
- may count "units" in order to compare two things but be fairly casual in their repetition of units, not noticing gaps or overlaps; e.g., placing the first "unit" away from the end when measuring length, not worrying about spills when measuring how much a container holds, not stopping their claps immediately the music stops
- do not necessarily expect the same "answer" each time when deciding how many fit
- many not think to use unit information to answer

The Diagnostic Map includes key indications and consequences of students' understanding and growth. This information is crucial for teachers making decisions about their students' level of understanding of mathematics. It enhances teachers' decisions about what to teach, to whom, and when to teach it.

Each developmental phase of the Diagnostic Map has three aspects. The first aspect describes students' major preoccupations *during that phase*. At the centre of each phase is the learning focus *during that phase*. This learning results in typical thinking and behaviour patterns *by the end of that phase*. Preconceptions, partial conception, or misconceptions, however, may still exist for students at the end of the phase. This final aspect provides the learning challenges and teaching emphases *as students move to the next phase*.

Diagnostic Tasks

First Steps in Mathematics: Measurement provides a series of short, focused Diagnostic Tasks in the *Course Book*. These tasks have been validated through extensive research with students and help teachers locate individual students on the Diagnostic Map.

How Do Teachers Advance This Learning?

To advance student learning, teachers identify the big mathematical ideas or key understandings of the outcomes or curricular goals. Teachers plan learning activities to develop these key understandings. As learning activities provide students with opportunities and support to develop new insights, students begin to move to the next developmental phase of mathematical thinking.

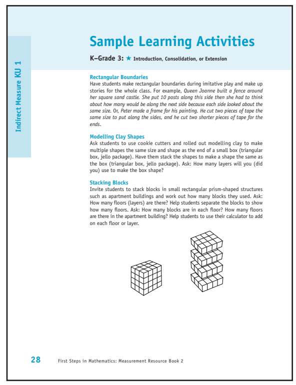
Key Understandings

The Key Understandings are the cornerstone of *First Steps in Mathematics*. The Key Understandings:

- describe the mathematical ideas, or concepts, which students need to know in order to achieve curricular goals
- explain how these mathematical ideas form the underpinnings of the mathematics curriculum statements
- suggest what experiences teachers should plan for students so that they move forward in a developmentally appropriate way
- provide a basis for the recognition and assessment of what students already know and still need to know in order to progress along the developmental continuum and deepen their knowledge
- indicate the emphasis of the curriculum at particular stages
- provide content and pedagogic advice to assist with planning the curriculum at the classroom and whole-school levels

The number of Key Understandings for each mathematics curricular goal varies according to the number of "big mathematical ideas" students need to achieve the goal.

Sample Learning Activities



For each Key Understanding, there are Sample Learning Activities that teachers can use to develop the mathematical ideas of the Key Understanding. The activities are organized into three broad groups:

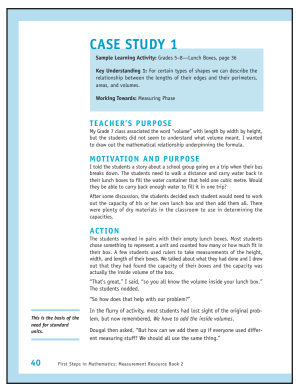
- activities suitable for students in Kindergarten to Grade 3
- activities for students in Grades 3 to 5
- activities for students in Grades 5 to 8

If students in Grades 5 to 8 have not had enough prior experience, then teachers may need to select and adapt activities from earlier groups.

Case Studies

The Case Studies illustrate some of the ways in which students have responded to Sample Learning Activities. The emphasis is on how teachers can focus students' attention on the mathematics during the learning activities.

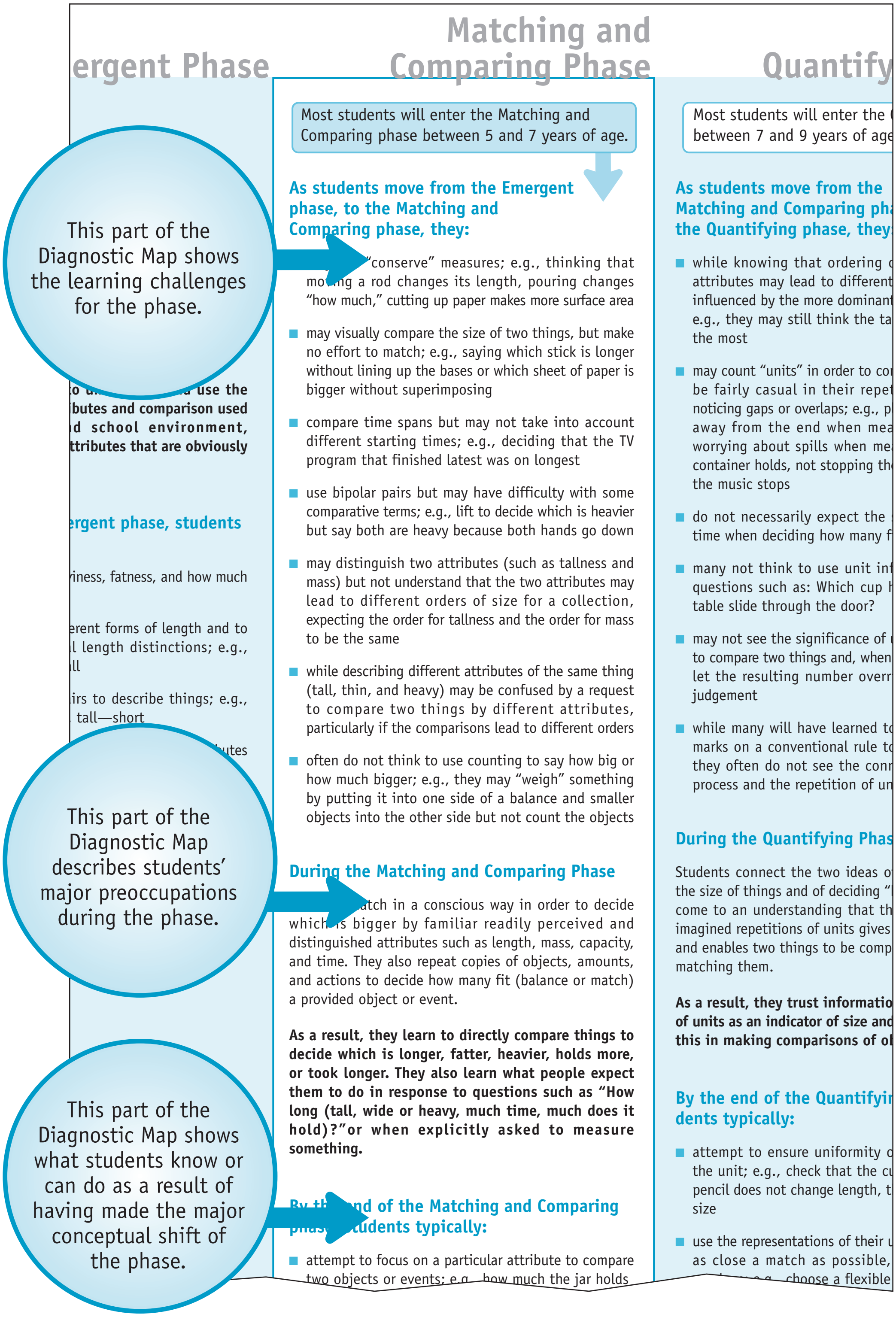
"Did You Know?" Sections



For some of the Key Understandings, there are "Did You Know?" sections. These sections highlight common understandings and misunderstandings that students have. Some "Did You Know?" sections also suggest diagnostic activities that teachers may wish to try with their students.

How to Read the Diagnostic Map

The Diagnostic Map for Measurement has five phases: Emergent, Matching and Comparing, Quantifying, Measuring, and Relating. The diagram on this page shows the second phase, the Matching and Comparing phase.



Matching and Comparing Phase

Most students will enter the Matching and Comparing phase between 5 and 7 years of age.

As students move from the Emergent phase, to the Matching and Comparing phase, they:

- "conserve" measures; e.g., thinking that mo ng a rod changes its length, pouring changes "how much," cutting up paper makes more surface area
- may visually compare the size of two things, but make no effort to match; e.g., saying which stick is longer without lining up the bases or which sheet of paper is bigger without superimposing
- compare time spans but may not take into account different starting times; e.g., deciding that the TV program that finished latest was on longest
- use bipolar pairs but may have difficulty with some comparative terms; e.g., lift to decide which is heavier but say both are heavy because both hands go down
- may distinguish two attributes (such as tallness and mass) but not understand that the two attributes may lead to different orders of size for a collection, expecting the order for tallness and the order for mass to be the same
- while describing different attributes of the same thing (tall, thin, and heavy) may be confused by a request to compare two things by different attributes, particularly if the comparisons lead to different orders
- often do not think to use counting to say how big or how much bigger; e.g., they may "weigh" something by putting it into one side of a balance and smaller objects into the other side but not count the objects

During the Matching and Comparing Phase

…atch in a conscious way in order to decide which is bigger by familiar readily perceived and distinguished attributes such as length, mass, capacity, and time. They also repeat copies of objects, amounts, and actions to decide how many fit (balance or match) a provided object or event.

As a result, they learn to directly compare things to decide which is longer, fatter, heavier, holds more, or took longer. They also learn what people expect them to do in response to questions such as "How long (tall, wide or heavy, much time, much does it hold)?" or when explicitly asked to measure something.

By the end of the Matching and Comparing phase, students typically:

- attempt to focus on a particular attribute to compare two objects or events; e.g., how much the jar holds

The text in the "During the phase" section describes students' major preoccupations, or focuses, during that phase of thinking about Measurement.

The "By the end" section of each phase provides examples of what students typically think and are able to do as a result of having worked through the phase.

The achievements described in the "By the end" section should be read in conjunction with the "As students move" section. Together, these two sections illustrate that although students might have developed a range of important understandings as they passed through the phase, they might also have developed some unconventional or unhelpful ideas. Both of these sections of the Diagnostic Map are intended as a useful guide only. Teachers will recognize more examples of similar thinking in the classroom.

How Do Students Progress Through the Phases?

Students who have passed through one phase of the Diagnostic Map are entering the next phase. They bring behaviours and understandings from one phase to the next. For example, the text in the "As students move from the Emergent Phase" section describes the behaviours students bring to the Matching and Comparing phase. This section includes the preconceptions, partial conceptions, and misconceptions that students may have developed along the way. These provide the learning challenges for the next phase.

Linking the Diagnostic Maps and Learning Goals

Students are unlikely to achieve full conceptual understanding unless they have moved through certain phases of the Diagnostic Map. However, passing through the phase does not guarantee that the concept has been mastered. Students might have the conceptual development necessary for deepening their understanding, but without access to a classroom program that enables them to learn the necessary foundation concepts described in a particular phase, they will be unable to do so.

The developmental phases help teachers interpret students' responses in terms of pre- and partial conceptions. If, for example, a student believes that large objects weigh more (have a greater mass) than small objects, then the phases can help explain what the problem might be. In this case, a student might not "conserve" mass, and no amount of telling the students that bigger things are not always heavier will help. The student needs multiple experiences matching and comparing objects of different density, first holding the objects, then placing the objects on balance scales. In this manner students will begin to change their incorrect perceptual understanding of bigger always being heavier—which was generally correct, so they trusted it was always correct—to the more reliable act of testing before predicting.

How Will Teachers Use the Diagnostic Map?

The Diagnostic Map is intended to assist teachers as they plan for mathematics teaching and learning. The descriptions of the phases help teachers make informed decisions about students' understandings of the mathematical concepts. The map will help teachers understand why students can do some things and not others, and why some students may be having difficulty achieving certain outcomes.

Initially, teachers may use the Diagnostic Map to extend their own knowledge about how students typically learn mathematics. Knowing about the major conceptual shifts in each phase and their links to mathematical learning goals will help teachers decide which Key Understandings should be the major focus for classroom planning.

Familiarity with the behaviours described in the phases will enhance the informed decisions teachers make about what they observe students doing and saying during lessons. The information obtained over time about the major preoccupations of students informs ongoing planning. As teachers begin to understand the typical behaviours of each phase, this planning process will become more efficient.

Planning with *First Steps in Mathematics*

Using Professional Decision-Making to Plan

The *First Steps in Mathematics* resource books and professional development support the belief that teachers are in the best position to make informed decisions about how to help their students achieve conceptual understanding in mathematics. Teachers will base these decisions on knowledge, experience, and evidence.

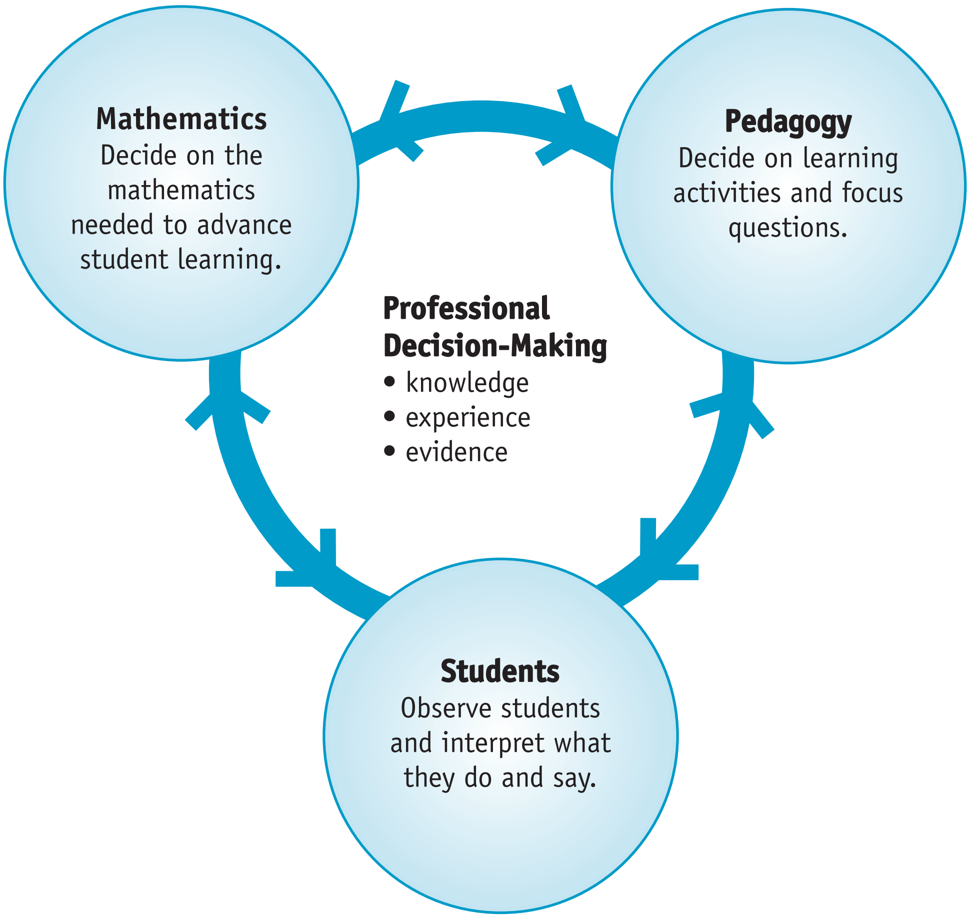
The process of using professional decision-making to plan classroom experiences for students is fluid, dependent on the situation and context, and varies from teacher to teacher. The selection of learning activities and appropriate focus questions will be driven by each teacher's knowledge of his or her students and their learning needs, the mathematics, and mathematics-related pedagogy. The *First Steps in Mathematics* resource books and professional development focus on developing this pedagogical content knowledge.

The diagram on the next page illustrates how these components combine to inform professional decision-making. There is no correct place to start or finish, or process to go through. Circumstances and experience will determine both the starting point and which component takes precedence at any given time.

Different teachers working with different students may make different decisions about what to teach, to whom, when and how.

The process is about selecting activities that enable all students to learn the mathematics described in curriculum focus statements. More often than not, teachers' choice of activities and focus questions will be driven by their knowledge of their students and the mathematics. At other times, teachers might select an activity to help them assess students' existing knowledge or because of the specific mathematics in the task. Whatever the starting point, the *First Steps in Mathematics* resource books and professional development will help teachers to ensure that their mathematics pedagogy is well informed.

The examples on the opposite page show some of the different ways teachers can begin planning using *First Steps in Mathematics*.



Focusing on the Mathematics

Teachers may choose to focus on the mathematics, deciding on the mathematics they think they need to move students on.

What mathematics do my students need to know?

Mathematics
Decide on the mathematics needed to advance student learning.

What sections of ***First Steps in Mathematics*** do I look at?

- Key Understandings and Key Understandings descriptions

Understanding What Students Already Know

Teachers may choose to start by finding out what mathematics their students already know.

What do my students know about these mathematics concepts?

Students
Observe students and interpret what they do and say.

What sections of ***First Steps in Mathematics*** do I look at?

- Key Understandings and Key Understandings descriptions
- "Did You Know?" sections
- Diagnostic Map
- Diagnostic Tasks

Developing Students' Knowledge

Teachers may begin by planning and implementing some activities to develop their knowledge of students' learning.

What activities will help my students develop these ideas? How will I draw out the mathematical ideas from the learning activity?

Pedagogy
Decide on learning activities and focus questions.

What sections of ***First Steps in Mathematics*** do I look at?

- Sample Learning Activities
- Case Studies
- Key Understandings and Key Understandings descriptions

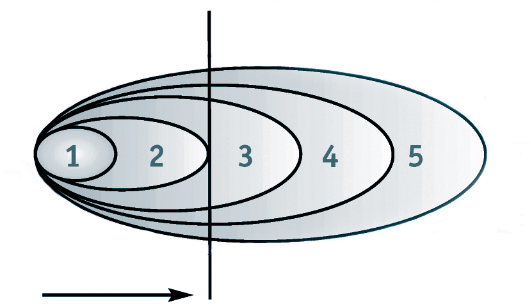
Planning

The mathematics curriculum goals and developmental phases described in the Diagnostic Map help teachers to know where students have come from and where they are heading. This has implications for planning. While day-to-day planning may focus on the mathematics of particular Key Understandings, teachers must keep in mind the learning required for progression through the developmental phases.

If a student has reached the end of the Matching and Comparing phase, then the majority of experiences the teacher provides will relate to reaching the end of the Quantifying phase. However, some activities will also be needed that, although unnecessary for reaching the Quantifying phase, will lay important groundwork for reaching the Measuring phase and even the Relating phase.

For example, students do not typically understand that they can partition a rectangle into appropriate squares and use the array structure to work out how many squares are in a rectangle, until approximately Grades 4-6 (ages 9–11). Therefore, understanding the significance of having no gaps and overlaps when "filling" rectangular arrays is not expected for reaching the end of the Quantifying phase, but it is for reaching the end of the Measuring phase. Given access to an appropriate program in Measurement, most students should be able to reach the Measuring phase, using an appropriate array structure without gaps to work out how many squares are in a rectangle, by Grade 6 (age 11). If students are to develop these ideas in a timely manner, however, the ideas cannot be left until after reaching the end of the Quantifying phase.

Developmental Phases



There are a number of reasons for this approach. First, it is expected that a considerable number of students will enter Grade 4 having reached the end of the Measuring phase. Second, if teachers are to wait until this time to start teaching about partitioning a rectangle into appropriate squares, having no gaps and overlaps, and using the array structure to work out the number of squares in a rectangle, then it is unlikely those students would develop all the necessary concepts and skills in a timely fashion. Third, work in Grades 4–6 should not only focus on the Measuring phase, but also provide the groundwork for students to reach the Relating phase in the next year or two, understanding that we can use length measurements to work out area.

Teachers, who plan on the basis of deepening the understanding of the concepts, would think about the expected long-term learning in the early years of schooling. They would provide experiences that lead to the learning goals at the Measuring and Relating phases. For example, in the *Same Number of Tiles* activity on pages 28 and 29, primary students are challenged to explore a concept—same area, different perimeter—that is delved into several years later. The students may not yet be ready to fully understand the significance of shapes with the same area possibly having different perimeters. It will take several years of learning experiences in a variety of contexts to culminate in a full understanding.

Monitoring Students over Time

By describing progressive conceptual development that spans the elementary-school years, teachers can monitor students' individual long-term mathematical growth as well as their long-term progress against an external standard. This long-term monitoring is one of the reasons why a whole-school approach is essential. For example, Sarah has reached the end of the Quantifying phase for each of the Measurement concepts while another student, Maria, has only just reached the Matching and Comparing phase.

By comparing Maria and Sarah's levels against the standard, their teacher is able to conclude that Sarah is progressing as expected, but Maria is not. This prompts Maria's teacher to investigate Maria's thinking about Measurement and to plan specific support.

However, if two years later, Sarah has not reached the end of the Relating phase while Maria has reached the end of the Measuring phase and is progressing well towards reaching the Relating phase, they would both now be considered "on track" against an external standard. Sarah's achievement is more advanced than Maria's, but in terms of individual mathematical growth, Sarah appears to have stalled. Her progress may now be of greater concern than Maria's.

Reflecting on the Effectiveness of Planned Lessons

The fact that activities were chosen with particular mathematical learning goals in mind does not mean that they will have the desired result. Sometimes, students deal with an activity successfully, but they use different mathematics than teachers anticipated. Different activities related to the mathematics that has not been learned may need to be provided in the future.

On other occasions, what students actually learn may not be what teachers intended them to learn. Students may surprise teachers and cause them to rethink the activity.

In some instances, activities, which teachers think will help students develop particular mathematical ideas, do not generate those ideas. This can occur even when students complete the activity as designed.

The evidence about what students are actually thinking and doing during their learning experiences is the most important source of professional learning and decision-making. At the end of every activity, teachers need to ask themselves: *Have the students learned what was intended for this lesson? If not, why not?* These questions are at the heart of improving teaching and learning. Teachers make constant professional, informed evaluations about whether the implemented curriculum is resulting in the intended learning goals for students. If it is not, then teachers need to change the experiences provided.

Teachers' decisions, when planning and adjusting learning activities as they teach, are supported by a clear understanding of:

- the desired mathematics conceptual goal of the selected activities
- what progress in mathematics looks like
- what to look for as evidence of students' deepening understanding

When planning day-to-day lessons, it is important for teachers to appreciate that many of the same activities will be appropriate for students who are working within a range of developmental phases. Teachers can accommodate the differences in understanding and development among students by:

- asking different questions of individual students and groups of students
- providing extension activities for selected students
- giving particular students opportunities to do different things with the activities

Chapter 2

Indirect Measure

Select, interpret, and combine measurements, measurement relationships, and formulas to determine other measures indirectly.

Overall Description

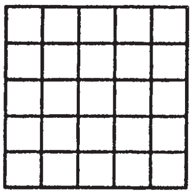
Indirect measurement is used when direct comparison or measurement of quantities is impossible, impractical, or simply tedious. Students choose and use a range of methods of indirect measurement. They weigh a few pieces of fruit at a time and add the weights because their scales will not accommodate more than 500 g. They predict when a video will finish by taking the time now and adding on the "length" of the film. They may also use division or averaging to find measurements more accurate than their equipment allows; for example, measuring the thickness of a ream of paper in order to calculate the thickness of one sheet. They also use formulas for finding lengths, areas, volumes, scale, and similarity; Pythagorean theorem and trigonometric ratios for finding lengths and distances in three dimensional contexts; and rates and derived measures such as speed and density for calculating quantities.

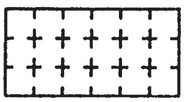
BACKGROUND NOTES

Structuring Rectangular Arrays

In the Background Notes for Direct Measure, the significance of an understanding of rectangular arrays for students' understanding of the way we measure area was highlighted. This is closely connected to the development of students' understanding of the use of formula for finding area.

Rectangles such as those in diagrams (a) to (k)[1] should be presented to students. Use your judgement about whether this should occur over time or in a concentrated period, depending on students' previous experience and present understanding. Each rectangle has its dimensions in centimetres. Begin with the rectangles that give the most graphic information about the subdivision of the rectangles and then gradually move to those that give less information. Give students a number of problems of each type of graphic representation, thus modelling the structuring process for them so that they can build their capacity to do it for themselves.


(a)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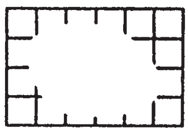
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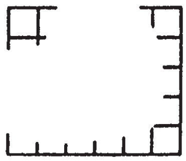
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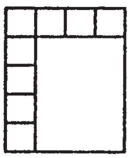
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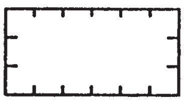
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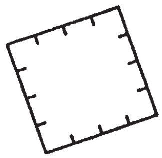
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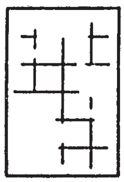
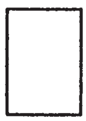

(h)


(i)


(j)
Students were shown that four square tiles go across the top (then the tiles were removed) and that three square tiles go down the middle (then the tiles were removed).


(k)
Students were shown that five square tiles go across the top (then the tiles were removed) and that seven square tiles go down the middle (then the tiles were removed).

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[1]Battista, M. 1999, The importance of spatial structuring in geometric reasoning, *Teaching Children Mathematics*, November, 170-177.

Activity Type 1

As you give a rectangle to students, help them see how a centimetre square fits on it. Have students first predict how many square centimetres will fit and then check their predictions with plastic or paper centimetre squares.

Activity Type 2

Later vary this, so that after students have made their predictions for the rectangles, have them draw how they think the squares will cover the rectangles, change their prediction if they wish, and then check with the squares.

Indirect Measure:
Key Understandings Overview

Teachers will need to plan learning experiences that include and develop the following Key Understandings (KU). These Key Understandings underpin achievement of this family of concepts. The learning experiences should connect to students' current knowledge and understandings rather than to their grade level.

Key Understanding		Description
KU1	For certain types of shapes we can describe the relationship between the lengths of their edges and their perimeters, areas, and volumes.	page 26
KU2	When two objects have the same shape: ■ matching angles are equal ■ matching lengths are proportional ■ matching areas are related in a predictable way ■ matching volumes are related in a predictable way	page 44
KU3	Scale drawings and models have the same shape as the original object. This can be useful for comparing and calculating dimensions and for making judgements about position.	page 58
KU4	We can calculate one measurement from others using relationships between quantities.	page 68

Grade Levels—Degree of Emphasis			Sample Learning Activities
K-3	3-5	5-8	
★★★	★★	★★	K-Grade 3, page 28 Grades 3-5, page 30 Grades 5-8, page 34
★	★★	★★	K-Grade 3, page 46 Grades 3-5, page 48 Grades 5-8, page 51
★	★★	★★	K-Grade 3, page 60 Grades 3-5, page 61 Grades 5-8, page 63
★	★★	★★★	K-Grade 3, page 70 Grades 3-5, page 71 Grades 5-8, page 73

Key

★★★ **Major Focus**
The development of this Key Understanding is a major focus of planned activities.

★★ **Important Focus**
The development of this Key Understanding is an important focus of planned activities.

★ **Introduction, Consolidation or Extension**
Some activities may be planned to introduce this Key Understanding, to consolidate it, or to extend its application. The idea may also arise incidentally in conversations and routines that occur in the classroom.

Key Understanding 1

For certain types of shapes we can describe the relationship between the lengths of their edges and their perimeters, areas, and volumes.

The focus of this Key Understanding is students' understanding of commonly used measurement formulas. For certain types of two dimensional shapes (e.g., rectangle, triangle, circle) we know the relationships between specified lengths and the perimeter. We also know the relationship between specified lengths and the area. For certain types of three dimensional shapes (e.g., rectangular prism, cylinder) we know the relationships between specified lengths and the surface area and volume. Formulas are a shorthand way of describing these relationships. The formulas are useful because they help us to work out perimeters, areas, and volumes more easily than measuring them directly.

Memorizing formulas is less important than understanding the relationships involved. Students need experiences over an extended period of time to understand these relationships. In particular, they will need to build up their understanding of the structure and use of rectangular arrays and how they link to multiplication (see Background Notes, page 22).

Students should investigate measurement relationships in a range of ways, developing their own short cuts for solving practical problems and investigating patterns in tables and graphs. For example, they could make a graph that shows the circumference of circular lids of various diameters. The points should, theoretically, lie on a line, but are unlikely to fit exactly because of measurement error. So long as the measurement is reasonably precise, however, the underlying relationship will still be evident and will enable students to predict the circumferences of other lids (and circles generally). Students should be encouraged to confirm shortcuts. A discussion and debate about shortcuts could be conducted to ensure student understanding and encourage shared understandings.

Links to the Phases

Phase	Students who are through this phase...
Quantifying	■ are able to find perimeters directly or by measuring edges and adding ■ are able to find the area of shapes by placing tiles on the shapes and counting, but are also beginning to predict how many tiles will cover a region by focusing upon the array structure of a rectangle and thinking about the numbers of rows
Measuring	■ will devise a plan and explain their own shortcuts for finding the perimeter of polygons *For example*: A student may measure one side of a regular penta-gon and multiply by five, or measure two adjacent sides of a rectangle, and add and double ■ understand that, although they could determine the area of a rectangle directly by covering it with unit squares and counting the number of squares and part squares, they could also work out the area of a rectangle composed of squares by thinking of it as an array and multiplying the number of squares high by the number of squares wide; that is, the number of rows by how many in each row ■ are able to build prisms from layers of cubes and can generalize about the relationships between the number of cubes along the sides and the total number of cubes in the shape *For example*: A student may have found that sometimes the tallest jar holds the most and sometimes it does not. The student may conclude that to get the most to drink, they need to focus on the capacity of the jar, not the height.
Relating	■ understand and use the formula for the area of a rectangle even when the side lengths are not whole numbers (which means that the rectangle cannot readily be thought of as an array and the relationship is no longer intuitive), and have learned to use this relationship to work out areas of other shapes *For example*: Are able to visualize a triangle as half of a rectangle and can rearrange a parellogram to form a rectangle of the same area. Each of these requires students to understand that if you join two regions or split a region, the total area will be the sum of the parts. ■ are able to use the formula for finding the volume of a prism from the length of its sides and break more complex shapes into prisms to find their volume

Sample Learning Activities

K–Grade 3: ★ Introduction, Consolidation, or Extension

Rectangular Boundaries

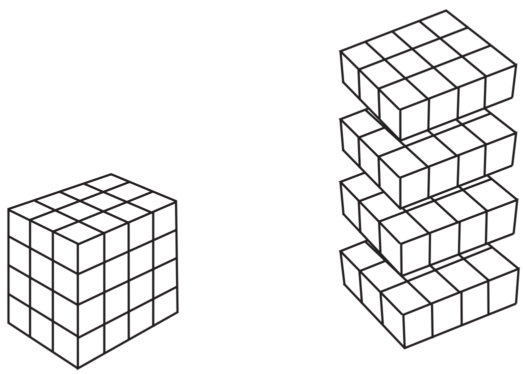
Have students make rectangular boundaries during imitative play and make up stories for the whole class. For example, *Queen Joanne built a fence around her square sand castle. She put 10 posts along this side then she had to think about how many would be along the next side because each side looked about the same size.* Or, *Peter made a frame for his painting. He cut two pieces of tape the same size to put along the sides, and he cut two shorter pieces of tape for the ends*.

Modelling Clay Shapes

Ask students to use cookie cutters and rolled out modelling clay to make multiple shapes the same size and shape as the end of a small box (triangular box, jello package). Have them stack the shapes to make a shape the same as the box (triangular box, jello package). Ask: How many layers will you (did you) use to make the box shape?

Stacking Blocks

Invite students to stack blocks in small rectangular prism-shaped structures such as apartment buildings and work out how many blocks they used. Ask: How many floors (layers) are there? Help students separate the blocks to show how many floors. Ask: How many blocks are in each floor? How many floors are there in the apartment building? Help students to use their calculator to add on each floor or layer.

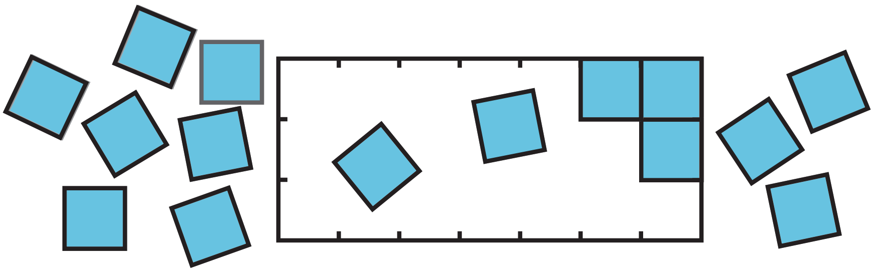


Same Number of Tiles

Have students make many different rectangle shapes with a given number of square tiles. Ask them to trace around their tiles to record each rectangle on paper and use the dimensions to describe their rectangles. For example, *I made this one with 6 rows of 2 tiles and this one with 3 rows of 4 tiles. I used 12 tiles each time*. Have students then measure the perimeter of their rectangles with string and compare the lengths. Ask: How many tile edges fit along the length of string? How is it that the two pieces of string are different lengths when you used the same number of tiles to make both rectangles? Why do you think that happens? Will it be different again if you make another 12-tile shape? Draw out the idea that even though there is the same number of tiles there are different distances around the shapes.

Covering a Rectangle

Invite students to work out the number of tiles taken to cover a rectangle. For example, give them multiple copies of a tile or square that exactly matches the markings around the rectangle. Invite them to try to arrange the tiles so they exactly cover the rectangle. Ask: How many squares does it take? How do the tiles fit with the marks around the edge of the rectangle? (See Background Notes, page 22, and Did You Know?, page 43.)



Sample Learning Activities

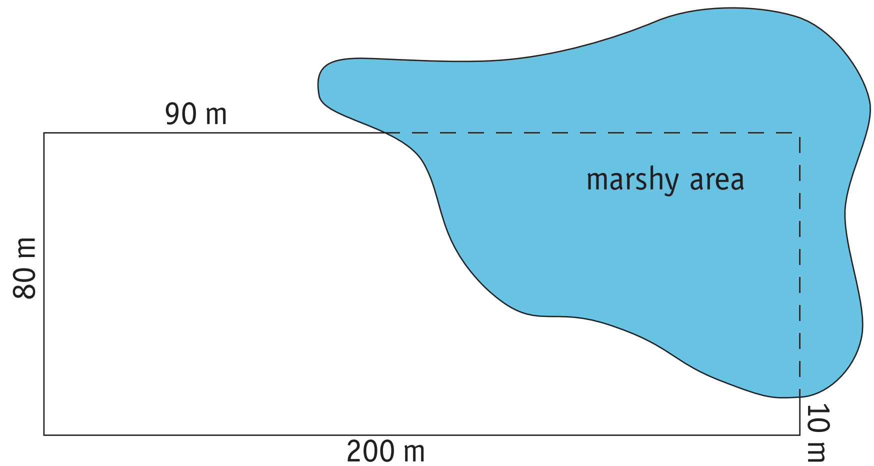
Grades 3–5: ★★★ Major Focus

Shortcuts for Perimeters

Have students directly measure the perimeter of various rectangles, then look for short cuts and write down instructions that others can follow. Invite them to share their methods and say how each is related to the sides of the rectangle. For example, *We measured the long side and the short side together and doubled the number.* Or, *We did it differently. We measured the long side, made it times two, and then added it to the short side times two*. Ask: Would your method work for all rectangles? Will it work for squares? Why? Why not?

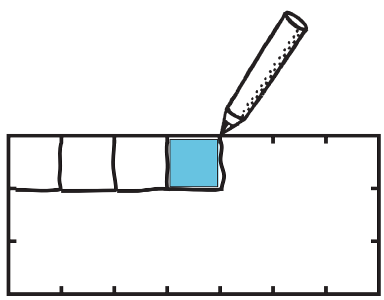
Fencing for a Pasture

Have students work out the perimeter of rectangles when part of the border is hidden. For example, say: The farmer wants to work out how much fencing to buy for his rectangular pasture but can only measure part of what he needs. Ask: Can you help him work out how much fencing he needs?



Covering a Rectangle

Extend K–Grade 3 Sample Learning Activity *Covering a Rectangle*, page 29, by asking students to first predict how many tiles they think will cover the rectangle. Invite students to use a cardboard square as a template and draw around it to check their prediction. Ask: How many squares cover the rectangle? How did you count them? How would counting how many squares in one row be helpful? How could your ruler help you with this? (rule lines to join the marks) (See Background Notes, page 22.)



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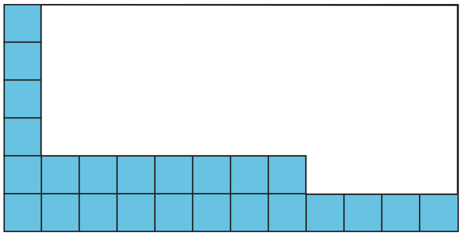
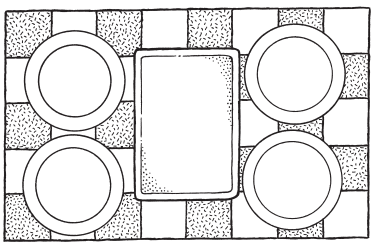
Incomplete Grids

Have students use incomplete grids to find the number of squares in a rectangle. For example, show students the diagram below and say: Amina made a table. She is covering it with decimetre square tiles. She has bought 54 tiles. Is this going to be enough to cover the table? How do you know? Does knowing that there are 12 in a row help you? How? Does knowing that there are six in a column help you to work it out? How?



Picnic Blankets

Have students use incomplete grids to find the number of squares in a rectangle. For example, say: Some students were working out how many squares there were on the picnic blanket. Could they work it out without taking the trays and plates off? How? How many squares are there in a row? How many squares are there in a column? How does knowing this help you work it out? Encourage students to share their counting, skip counting, adding or multiplying strategies and decide which one is the quickest and easiest.



Twenty-Four Tiles

Have students construct as many different rectangles as possible with 24 tiles, using all the tiles each time. Ask them to record each one on grid paper and in a table, for example:

Side 1	Side 2
3	8
2	12
24	1

Have them order the rectangles so that the length of side 1 increases as they read down. Ask: What do you notice about the pairs of numbers? (they multiply to give 24) Why does this happen? (the number in side 1 tells you how many tiles in one row and the number in side 2 tells you how many rows) How can we check we have made all the possible rectangles?

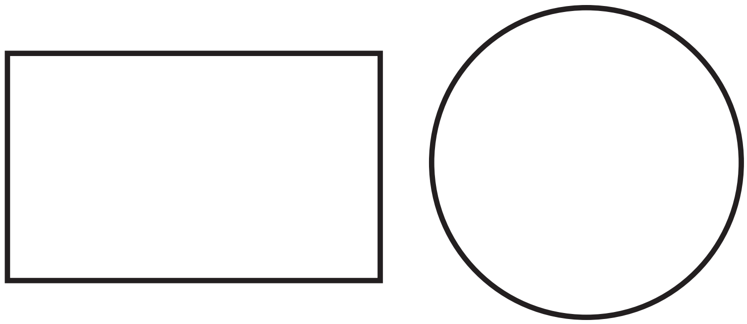
Grades 3–5: ★★★ Major Focus

Oil Spills

Give pairs of students an 11 × 17 copy of a map showing two different oil spills. Ask them to use squares of paper (2 cm^2) to compare the areas of the oil spills. Ask: How can we place the squares so that there are no gaps between them? Can you use rows of squares? Can you arrange the rows into rectangles that fit inside the shapes? How could you make the counting of the squares easier? Help the students to move from counting all the squares to counting how many in each row and adding the rows.

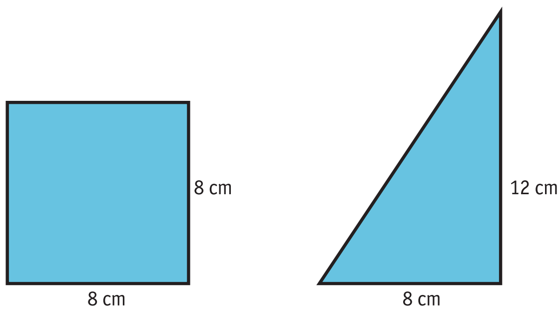
Pizza Trays

Have students carefully arrange square units in a range of shapes to find the area. For example, ask students to measure the area of circular and rectangular pizza trays to find out which is larger. Ask: Why is it easier to count the squares inside the rectangular shape? Can the squares in the circular shape also be placed into rows? How would that help you work out how many squares in the circle? What can you do about the gaps left around the edge?



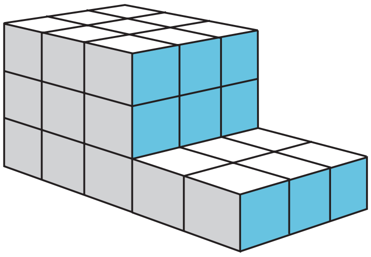
Which Pasture Is Bigger?

Have students use Base Ten plastic tiles to explore ways to work out which of the pastures shown below is bigger. Ask: Which is easier to measure? Why? Can you find a way to use the tiles for the triangle? How would arranging rows of tiles in an array help? Would it be useful to imagine creating a rectangle by joining a congruent triangle to the given triangle? How?



Buildings

Ask students to construct solid models of buildings with cubes, telling them that only full faces of cubes can touch each other. Have their partners work out how many cubes have been used. Ask: Can we use addition rather than counting each block separately? What could be added together? Can you see larger box shapes within the building? How can you use your calculator to help work out how many blocks are needed? Could multiplying help?



The Sealed Room

Have students use cubes to build rectangular prisms in order to work out the volume of the prisms. For example, say: People are locked in a sealed room 4 m long, 2 m wide, and 3 m high. We need to know how much oxygen there is, so we need to work out the volume of the room in cubic metres. Ask students to build the room with Base Ten cubes, pretending each is a metre cube. Ask: What is the volume of the prism? Then, say: What if they were locked in a sealed room 3 m long by 3 m wide and 2 m high? What would the volume be? Have students record their results in a table.

Length	Width	Height	Volume
4 m	2 m	3 m	24 m^3
3 m	3 m	2 m	18 m^3

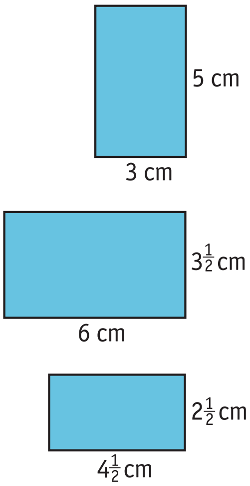
Invite students to explore other rooms with different dimensions and add the information to the table. Ask: Can you see any patterns? Can you use your calculator to find short cuts for working out the number of cubic metres? Could you work out the volume for another room without using the cubes?

Twenty-Four Cubes

Ask students to build rectangular prisms from 24 cubes each and record the dimensions in a table: how many cubes in a row, how many rows in one layer, and how many layers. Ask: How do you know you have made every possible rectangular prism? How could you work out the number of cubes in any rectangular prism from its measurements?

Sample Learning Activities

Grades 3–5: ★★★ Major Focus



Three Rectangles

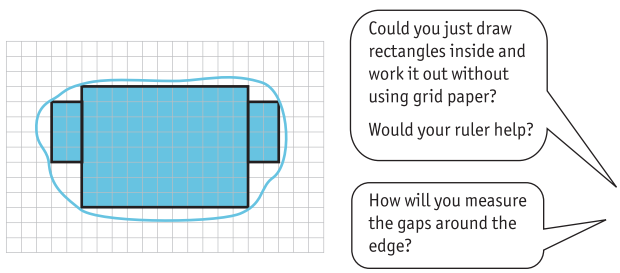
Give students a series of rectangles drawn to full size, but without the measurements shown. Include whole number and fraction examples. Ask them to work out the perimeter of each one. Ask: How did you work out the perimeters? Do all the ways of working it out give the same result? Should they? Why? Why not? Which ways were the easiest? Which were the most accurate?

Sod and Rope

Extend *Three Rectangles* above by saying: A gardener wants you to work out the perimeter and area of rectangular lawns for her because she needs to know how many square metres of sod to buy and how much rope she will need to enclose the lawns while the sod grows. Challenge students to find the easiest way to do this. Ask: Can you find rules that will work for every rectangular plot? What would you write down so that someone else would understand your rules? Are your rules the same or different from others? How?

Irregular Areas

Have students look for arrays within irregular shapes to help work out the area. For example, say: The farmer needs to find out the area of her pasture to know how much seed to buy. Ask: How could she work out the area? Could rectangles that fit inside the shape help with this? How? Invite students to draw the shape onto the grid paper to help them see the arrays within. Later, ask: Could you work it out without using grid paper?



Square Straws

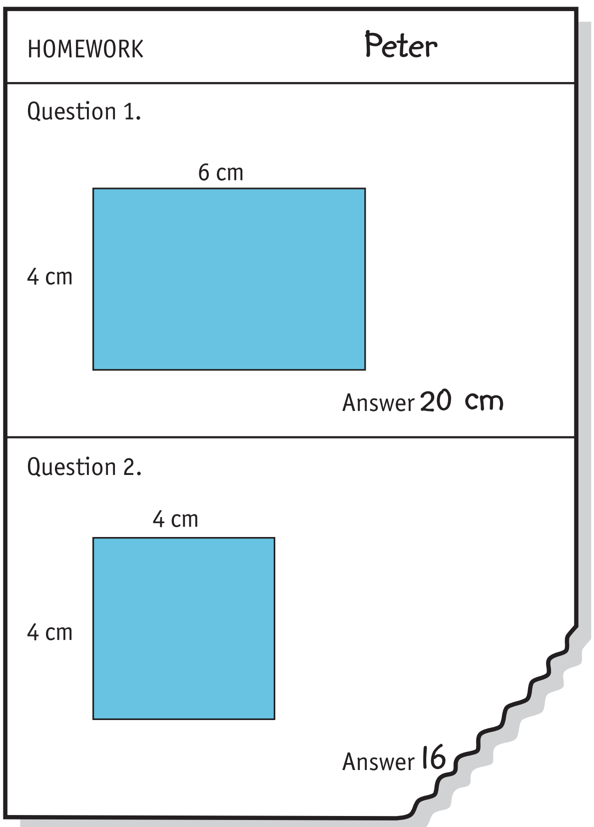
Ask pairs of students to make different-sized rectangles from straws (craft sticks, toothpicks). Invite them to describe the area of their rectangles. For example, *My rectangle is 2 straws by 4 straws and has an area of 8 square straws*. Draw out why the linear unit they chose needs to be squared. Invite students to give their partner an area measure (e.g., 10 square toothpicks) and have their partner make as many rectangles as they can with that area. Ask: Why do you need to know the unit before you can make the rectangle?

Dimension, Perimeter, or Area

Extend *Square Straws* above by having students calculate the areas and perimeters of rectangles where the dimensions are given in standard length units. Ask them to write their answers and explain why they needed to include units with the numbers, and how they chose which units to use. For example, ask: Is the area of a 4 cm by 3 cm rectangle 12, 12 squares, 12 cm, 12 square cm or 12 cm^2? What if the dimensions include fractions? How does that affect the area?

Perimeter or Area?

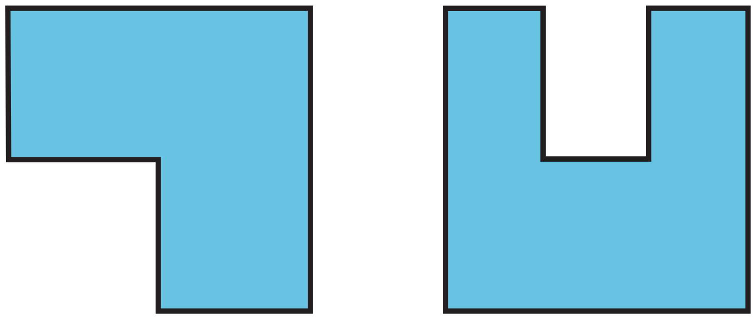
Have students decide whether perimeter or area measure has been used. For example, say: This piece of homework was found on the floor. Ask: What might the questions have been? How can you tell? What did you do to check? Why can you not say for sure whether question 2 is "What is the perimeter?" or "What is the area?" How would seeing the units tell us for sure? (cm or cm^2) Are there any other rectangles where the perimeter and the area both have the same number of units?



Grades 5–8: ★★★ Major Focus

House Plans

Have students use house plans drawn to a simple 1 cm to 1 m scale to work out the floor area of different-shaped rooms for a purpose (how much carpet or how many floor tiles would be needed to cover it). Ask: What can you do to work out the area of the rooms? How would thinking of the rooms as separate rectangles be helpful? What tools could you use to help? (graph paper, ruler, scissors) How would you use the tools? Once students have worked out the area, stimulate a class discussion by asking: Which ways were quicker and easier? Why?



Lunch Boxes

Have students use cubes to solve volume and capacity problems, looking for shortcuts to arrive at the total number of cubes used. For example, say: A school bus was carrying a group of students on an excursion when it broke down. The students had to walk a distance and carry water back in their lunch boxes to fill the water container that held one cubic metre. Would they be able to carry back enough water to fill it in one trip? Invite students to use cubes to work out the capacity of their lunch boxes and encourage them to look for shortcuts. Ask: How do the rows and layers of cubes help you think about how many cubes fit or match? How could you use your calculator to make it easier to count how many? How can you work out the volume if you do not have enough cubes to fill all the box? (See Case Study 1, page 40.)

The Sealed Room

Extend *The Sealed Room*, page 33, by having students build some larger rooms. Make sure there are not enough cubes to build the rooms completely. Ask: Can you work out how many cubes you would need if you completely built the room? Would counting rows and layers help? How? Invite students to work out a general rule that would work for any room. Ask: How can you be sure it will always work?

Cubic Straws

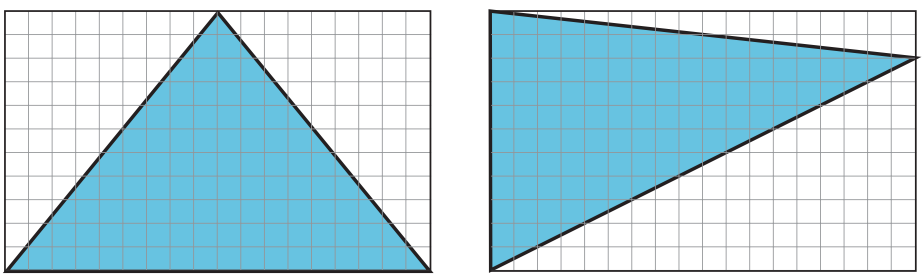
Extend *Square Straws*, page 34, by asking pairs of students to build rectangular prisms from straws (craft sticks, toothpicks) using modelling clay or joiners for the corners. Help them describe the volume of their prisms in terms of the linear unit used (cubic straws, cubic craft sticks, cubic toothpicks). For example, *My prism is 2 straws high, 2 straws across and 3 straws long and its volume is 12 cubic straws*. Ask: How is a 1-straw cube (1-craft stick cube, 1-toothpick cube) related to your prism? What do you mean when you say the volume is 12 cubic straws (craft sticks, toothpicks)? (*If I had 12 wooden cubes, each one straw wide, I could build a prism exactly the same size and shape as my straw skeleton*.) Why do we say cubic straws, rather than just straws when giving the volume? Could you make a prism that is not a cube, but has a volume of 1 cubic straw? What if you could cut some of your straws in half? What might the dimensions of the prism be? Later, have students use standard length units to describe the dimensions of rectangular prisms and to calculate and record volume.

Fractional Dimension

Extend *Lunch Boxes*, page 36, and *The Sealed Room*, page 36, to include some prisms with one fractional dimension. For example, say: A room is 3 m long, 4 m wide, and 2.5 m high. How can we work out the volume of the room? What would a half layer be like? Can your general rule still be used?

Triangle in a Rectangle

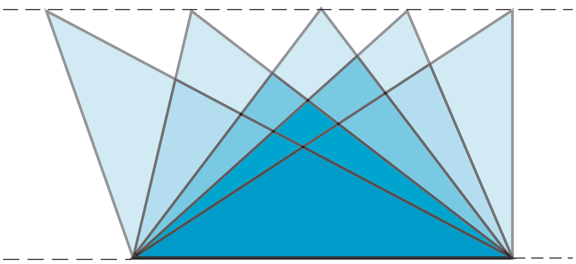
Have students draw a rectangle around a triangle. Invite them to compare the area inside the triangle to the area outside. Ask: What do you notice? Try other triangles that can be enclosed by the same-sized rectangle. Ask: Is the area outside the triangle the same as the area inside the triangle for others as well? What is the area of the rectangle? How could you use the area of the rectangle to easily work out the area of the triangles? Can you write down a general rule that others could use to work out the area of a triangle from the area of the rectangle it will fit inside? Have students use the general rules of other students to see if they work. Ask: Can you explain why the general rules work?

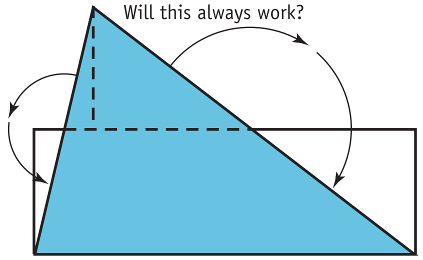


Grades 5–8: ★★★ Major Focus

Five Triangles

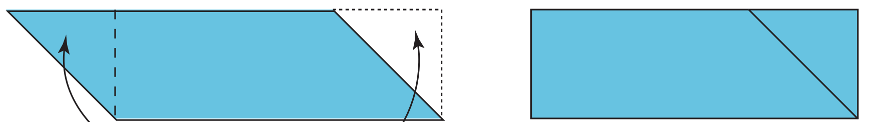
Have students investigate the areas of different triangles with the same height and base. For example, say: Jeremy has a new rule that says, *All triangles drawn on the same base and with the same height have the same area*. Ask: Does it work? Always? Must it? Can you find a reason why? How does it help you work out the area of triangles? Invite students to cut and rearrange the parts of each triangle so that it fits into a rectangle that has the same width as the triangle's base. Ask: Are all the rectangles the same height? Compare this to the height of the original triangles (should be half the height). How could you use what you know about the area of rectangles to work out the area of triangles?





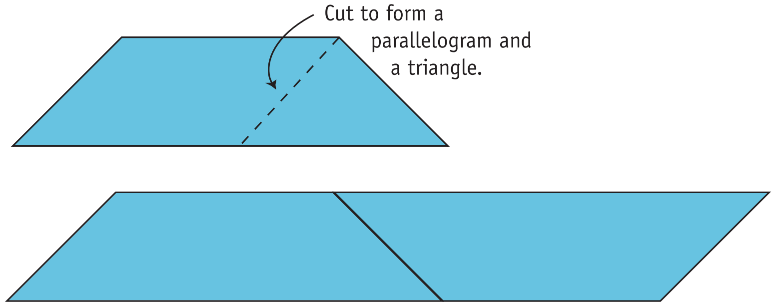
Rearranging Parallelograms

Have students cut and rearrange parallelograms to make rectangles. Invite them to investigate relationships between length measures on the parallelograms and on the related rectangles. Ask: Can you work out a formula (set of rules) that others can use to calculate the area of any parallelogram? Encourage students to use what they know about finding the area of a rectangle.



Rearranging Trapezoid

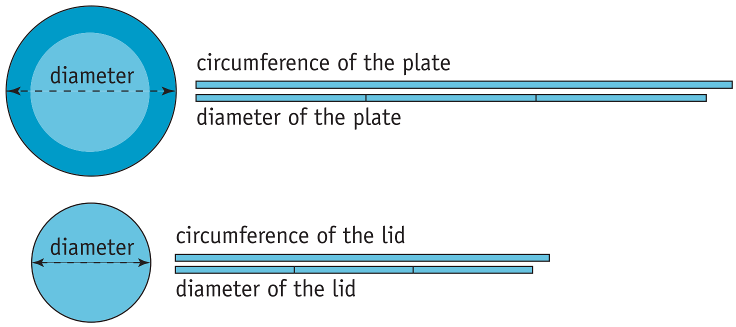
Extend Rarranging Parallellograms to discover the area of a trapezoid. Have students cut and rearrange trapezoids to create parallelograms and triangles. They can also rearrange two trapezoids to create a parallelogram with twice the area.



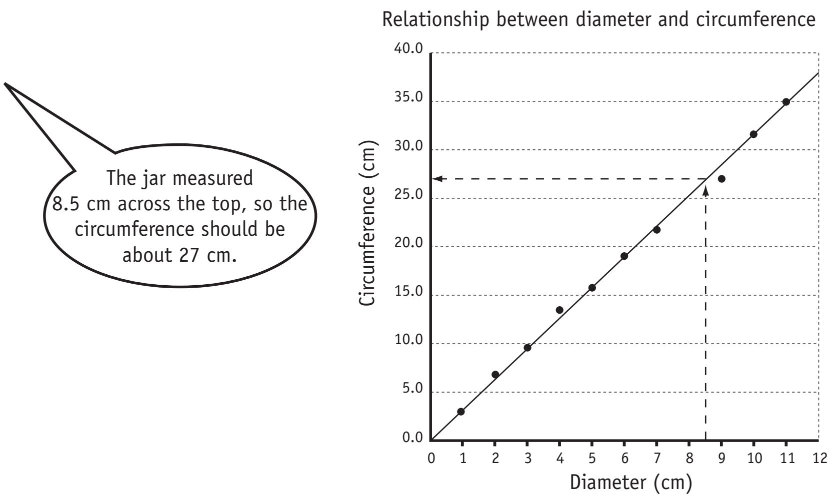
Circles

Ask students to find the diameter and circumference of circular objects (lids, plates, wheels) with string or thin tape. Have them display their results by placing diameter lengths beside the circumference length for each object.

Ask: What do you notice about the number of diameters in each case? (the circumference is just over three diameters)



Have students measure their strips and record the diameter and circumference for each object. Combine the information and draw a class graph of all the results with diameter on the horizontal axis and circumference on the vertical axis. Ask: Can you see a pattern in the points? (Ideally the points would be exactly on a line, but in practice students' measurements will often vary a little.) Draw out that the points are fairly close to being on a line. Draw the line. Invite students to measure the diameter of a different circle. Use the graph to estimate its circumference. Ask: How could you check this on the circle? How could you use your calculator to check the result?



Fringing

Extend *Circles* above by asking students to use the relationship that the circumference is three and a bit times the diameter to estimate quantities. For example, say: The diameter of a circular lampshade (cushion) is 35 cm. How much fringe (braiding) will I need to go around its edge?

CASE STUDY 1

Sample Learning Activity: Grades 5–8—Lunch Boxes, page 36

Key Understanding 1: For certain types of shapes we can describe the relationship between the lengths of their edges and their perimeters, areas, and volumes.

Working Towards: Measuring Phase

TEACHER'S PURPOSE

My Grade 7 class associated the word "volume" with length by width by height, but the students did not seem to understand what volume meant. I wanted to draw out the mathematical relationship underpinning the formula.

MOTIVATION AND PURPOSE

I told the students a story about a school group going on a trip when their bus breaks down. The students need to walk a distance and carry water back in their lunch boxes to fill the water container that held one cubic metre. Would they be able to carry back enough water to fill it in one trip?

After some discussion, the students decided each student would need to work out the capacity of his or her own lunch box and then add them all. There were plenty of dry materials in the classroom to use in determining the capacities.

ACTION

The students worked in pairs with their empty lunch boxes. Most students chose something to represent a unit and counted how many or how much fit in their box. A few students used rulers to take measurements of the height, width, and length of their boxes. We talked about what they had done and I drew out that they had found the capacity of their boxes and the capacity was actually the inside volume of the box.

"That's great," I said, "so you all know the volume inside your lunch box." The students nodded.

"So how does that help with our problem?"

__This is the basis of the need for standard units.__

In the flurry of activity, most students had lost sight of the original problem, but now remembered, *We have to add the inside volumes*.

Dougal then asked, "But how can we add them up if everyone used different measuring stuff? We should all use the same thing."

All the students agreed that this was a good idea. There was then some heated discussion about what to use, with some students favouring materials that could be poured, such as rice, because they were easy to use and others favouring cubes because they stacked. Finally, to my relief, the cube brigade won, using the argument that even if the students worked out how many scoops of rice there were altogether, they would not know how many scoops was equal to a cubic metre.

CONNECTION AND CHALLENGE

But still there was a problem. There was not enough of any one material for everyone to use! The challenge was to come up with a way to find the volume inside their lunch boxes without having enough cubes to fill it.

Most students used up all their blocks and then looked for ways of calculating how many more would fit.

I noticed that quite a number of pairs had carefully counted the number in the bottom. I stopped the class and asked Dougal and Olivia to explain what they were doing. Dougal said that they had 220 on the bottom and thought that each layer of cubes would be the same so they just needed to work out how many layers. I asked the students how many others were trying that approach and a number were.

"How will you work out how many layers?" I asked.

"By seeing how many go up the side," several students suggested.

"So, Olivia, how many go up the side of your lunchbox?" I asked.

"Five," she said. I noticed that Olivia actually needed six layers and had not counted the bottom layer in her five going up the side.

"So, what will you do with the five?" I asked

Olivia said she would add on five more lots of 220.

I asked the students to help Olivia and work out the volume for her. The class all agreed that 1320 cubes would fit in her lunch box. "Is there a shortcut for that?" I asked.

"You could multiply," several students replied.

I asked them to multiply and, of course, some multiplied by five while others multiplied by six, so that when I asked what the answer was, there were two different responses. I left it to one of the confident students to explain why you needed to multiply by six.

"There are six lots altogether, six lots of 220."

"So, what does the 220 tell you, Olivia?" I asked.

"How many in the bottom," Olivia said.

"And what does the six tell you?" I asked.

"How many lots of 220," she replied.

I had previously decided that, if the majority favoured a material such as rice, they would use it. There would not have been enough material and so students would have run into trouble when they tried to measure all the lunch boxes. I thought I might use that to persuade them to use cubes. If, however, they came up with a good strategy using their chosen material, I would have returned to cubes in a follow-up lesson. That the students decided to use 1-cm cubes simply meant we reached the point I wanted more quickly.

DRAWING OUT THE MATHEMATICAL IDEA

I rephrased Olivia's responses. "Olivia has said that the 220 is how much in a layer and the six is the number of layers." I then asked, "So, if we want to find the inside volume of our boxes, what do we need to do?"

Jodie replied, "Find out how many cubes it takes to cover the bottom and then how many layers of cubes you would need to fill it and then multiply them."

I wrote this on the board to give it emphasis and beside it I wrote:

V = number in one layer × number of layers

Students who had filled more than one layer began moving blocks into a stack up the side. I had noticed earlier that Hanadi and Justin had placed cubes inside their lunch box along two adjoining edges then made a stack in a corner. After most students had worked out the volume of their two boxes using the layers, I asked Hanadi to explain their approach.

"We only had to multiply the rows together, then multiply the answer with the height."

"Would this work for all of the lunch boxes?" I asked. "Test it on your boxes."

Students reached for their calculator to do the multiplication.

"Yes, it does," agreed most.

"So, why does it work for all of the different boxes?" I asked.

"Well," said Hanadi, "you are really finding out how many in the bottom layer and then how many layers you have." Others agreed. "Yes, you just multiply how many in one layer by the number up the side."

"What does this have to do with how many in a layer?" I asked.

Dougal offered, "You are just multiplying the sides." There were a number of nods of agreement—the class had previously found areas of rectangles based on square grids.

I wrote on the blackboard:

V = <u>side 1</u> × <u>side 2</u> × number of layers

and asked them to tell me again what the underlined parts showed.

"Hey," said Ariel, "that is like volume is equal to length times width times height that we did last year."

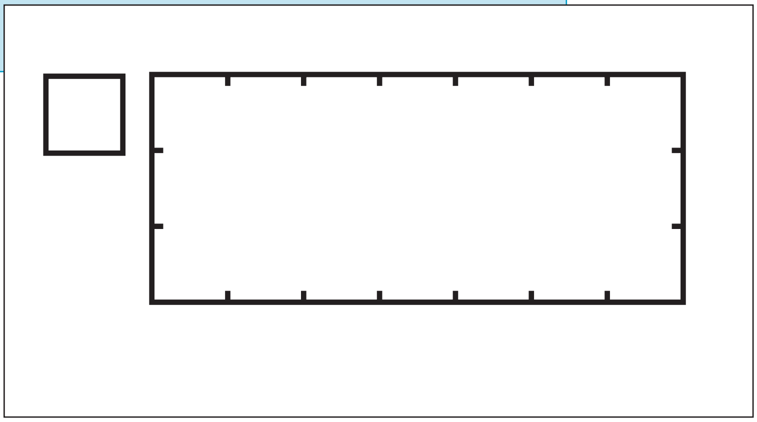
"Yes," I said, "it is the same. Side 1 could be the length and side 2 could be the width. The volume of a rectangular box can always be worked out by multiplying the length of the two sides by the height of the box."

The students then used this method to work out the inner volumes of their lunch boxes, which were recorded on the board and then added by the students. At this point, several students realized that they had worked out the volume in cubic centimetres and had to decide whether they had more or less than a cubic metre. The lesson continued.

Did You Know?

Show students a rectangle marked as shown below and a square cut out that exactly matches the markings so that, for this example, seven squares fit across the rectangle and three fit down. Ask them to predict how many squares they need to cover the rectangle. Do not initially give students a square to place or draw around, although later they should check their prediction.

We often assume that students can visualize the arrays in rectangular arrangements, but a surprising number have difficulty with tasks such as this.



Key Understanding 2

When two objects have the same shape:

- matching angles are equal
- matching lengths are proportional
- matching areas are related in a predictable way
- matching volumes are related in a predictable way

The focus of this Key Understanding is the development of students' understanding of what we mean mathematically when we say that two figures or two objects are the same shape and of some of the basic mathematical relationships involved (see Background Notes, page 22).

When we use a photocopier to enlarge or reduce something, the essential idea is that the copy should look the same as the original. Thus, the shape of the copy must be the same as the shape of the original. This is achieved by ensuring that angles on the copy are the same as matching angles on the original, and that lengths on the copy are a fixed multiplier of matching lengths on the original. This fixed multiplier is called by different names in different contexts: the scale factor, scale ratio, enlargement factor, magnification. If we want to double the dimensions of the original, we use a scale factor (multiplier) of × 2 (or 200%). Every length of the original is then doubled, while the angles are kept the same. If we want to halve the dimensions of the original, we enter × 0.5 or 50% of the original, thus setting the photocopier to halve all lengths. If we want it to be one-and-a-quarter times as big, we enter 1.25 or 125%. If the scale factor is bigger than 1, the copy (or scaled version) will be bigger; if the scale factor is less than 1, the scaled version will be smaller.

When we produce a copy, we are sometimes surprised at the size of what we produce. For example, when we make a half-sized copy of the word dog, using a 50% scale, the copy we produce may seem much smaller than half. This is because the area of the copy will be one quarter the size and so look much less than half the size.

dog

dog

The effect can be even more surprising for a three dimensional object. When we halve its dimensions, all areas are reduced to one quarter, but the volume is reduced to one eighth.

Activities should be provided that help students to develop an understanding that when we enlarge or reduce figures and objects we change the size without changing the shape. This means that the angles do not change, but all the lengths change by the same multiple (called the scale factor). Older students should begin to investigate the effect of scale changes (e.g. tripling all the length dimensions) on the perimeter, the area, and the volume of shapes. This will lead, during grades 7 and 8, to the generalization that, if two objects have the same shape:

- each angle on the first will be equal to the matching angle on the second
- each length on the first will be a fixed multiple (say, times k) of the matching length on the second
- each area on the first will be k^2 times the matching area on the second
- each volume on the first will be k^3 times the matching volume on the second

Links to the Phases

Phase	Students who are through this phase...
Quantifying	■ show a general sense of scale when selecting things for a purpose ■ may adjust items for a model they are building *For example*: A student may say, "It needs to be smaller to look right."
Measuring	■ use grids to enlarge and reduce in specified ways to produce systematic distortions ■ understand that for a copy to look the same as an original all lengths must be multiplied or divided by the same amount (for example, all halved or all tripled) and angles must remain the same ■ are able to predict the length of lines on the copy from the length of lines on the original
Relating	■ are able to use grids and arrangements of cubes to investigate and draw conclusions about the effect of scaling linear dimensions on the perimeter, area and volume of figures and objects ■ work out that if an arrangement of four cubes is scaled up by a factor of three (that is, made three times as big in each direction), then 27 times as many cubes will be needed; that is, 108 cubes

Sample Learning Activities

K–Grade 3: ★ Introduction, Consolidation, or Extension

Enlarging

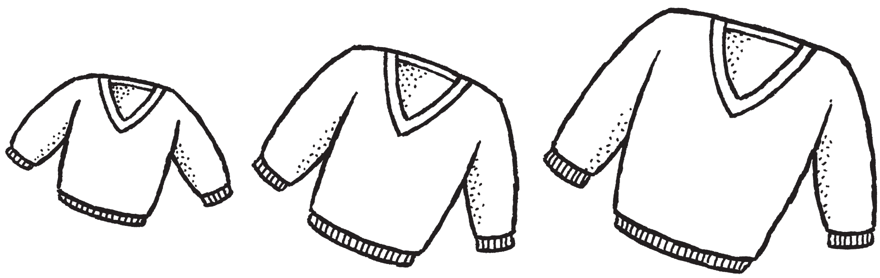
Have students use small square tiles to create a shape or design. Then, have them copy the shape or design, making each dimension twice the size, by encouraging them to first look at one tile and make another shape that is two tiles wide. Ask: How many tiles will you need to get it to be a square? Why are two tiles not enough? (to make it square you have to double the height as well) Encourage students to look at enlarging their design by making every unit square into a bigger square that is two squares wide and two squares high.

Mothers and Babies

Show students mother and baby animal pictures and have them describe the differences between them. Ask: Is the baby an exact copy of the mother? Which parts of the baby's body will change the most? Which baby animals are almost the same shape as their parents?

Sweaters and Chairs

Have students compare objects that vary in size (e.g., a baby's sweater, a child's sweater and an adult's sweater; a Grade 1 chair, a Grade 4 chair and a Grade 7 chair). For example, ask: How are the sweaters almost the same? How can you tell they are all sweaters? (the shape is almost the same) How are the sweaters different?



Graduated Sets

Have students use graduated sets of objects for imitative free play (sets of mixing bowls, sets of pots and pans, plates, measuring cups, measuring spoons, nesting cups, baskets). Mix the objects together and ask students to sort them into sets to pack away. Focus the discussion on the fact that the items are almost the same shape but different sizes. Ask: How do you know all the bowls go together? This pot is the same size as this bowl, so why not put it with the bowls? What is the same about all the bowls?

Craft Stick Squares

Have students make a square using four craft sticks. Ask them to make a large copy of the square using two or three craft sticks for each side. Display the range of sizes. Ask: Are the shapes the same (different)? How? What did James do to the first square to make this square? Did he do that to only one side? Why? Extend this to other rectangles to draw out the idea that to keep the shape every side must be changed in the same way (e.g., if one side is doubled, all sides must be doubled).

Photocopy Enlargements

Cut 8-1/2 × 11 sheets of paper into quarters and have groups of students draw some pictures on the quarter sheets. Use the photocopier to enlarge each twice. Return the original and the two enlargements to the students and ask them to compare the pictures in their groups. Ask: What has changed? What has stayed the same?

Sample Learning Activities

Grades 3–5: ★★ Important Focus

Triangles and Other Shapes

Have students use pattern blocks to explore the effect of enlarging by doubling and tripling dimensions. For example, use the triangles to make a larger triangle with each side twice as long as a single triangle. Ask: How many triangles did you need to make the sides of the triangle twice the size? Was it twice as many? Why not? Ask students to make a shape that is three times the size of the single triangle. (a trapezoid) Ask: What part of the shape is three times as big? Then ask students to make a shape with all sides three times as big as all sides of the triangle. Ask: How many triangles did you use? Was it three times as many? Why not? How many times as many was it? Invite students to work out how the area measures change when they double and triple the side lengths of the blue rhombus and the square.

Enlarging a Design

Have students use straight lines to draw a design on a four-by-five grid that has 1-cm squares. Then have them make a copy on another four-by-five grid that has 2-cm squares. Invite students to compare the lengths (including diagonals), areas, and angles. Ask: What has changed? What has stayed the same? How have the lengths changed? How has the area changed? Later, include designs with curved lines. Ask: How has the length of the curved lines changed? How has the area changed?

Reducing a Design

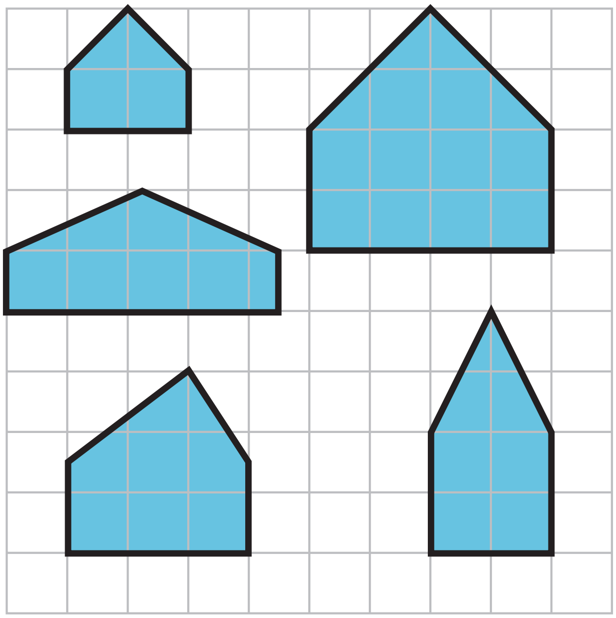
After activities like *Enlarging a Design* above, ask students to say what changes when the straight-line and curved-line designs are reduced. Ask: If the length of the straight lines are half as long, what do you think will happen to the length of the curved lines? How has the area changed?

Comparing Enlargements and Reductions

Have students compare their enlargements and reductions from *Enlarging a Design* and *Reducing a Design* above. Ask: Are the changes to each measurement the same (different)? How? Draw out that by doubling the lengths, the area is always multiplied by four. If the lengths are halved, the area is one quarter.

Figures on a Grid

Give students figures on a grid that are enlargements or distortions of each other. Have them circle the enlargement that is the same shape but bigger (smaller). Discuss reasons for their choices. For example, *That one is not the same shape, that one has been stretched and that one has been squashed*. Use a variety of simple, everyday shapes and ask students to decide which ones are the same shape but bigger (smaller).



Graphics

Invite students to use a graphics software to create images. Ask them to predict how the image will change when it is dragged from a corner or from the horizontal or vertical edge. Ask: What measurements are changing when it is dragged from the top (side, corner)? Is it an enlargement (reduction) or a distortion?

Making Cubes

Invite students to create a cube using plastic interlocking squares or squares of card taped together. Then, have them create another cube that is "twice the size." Ask: What does twice the size mean? Draw out the ambiguity. Ask: Is it a cube with sides twice as long? Is it a cube that takes up twice as much space? Say: Suppose we mean we want a cube with sides twice as long. How many pieces were needed for the original cube? How many will be needed for the bigger cube? Why did we need four times as many instead of just twice as many?

Grades 3–5: ★★ Important Focus

Wooden Cubes

Repeat *Making Cubes*, page 49, using wooden cubes. Invite students to put out one cube and then make another cube that is twice the size of the first. Invite students to predict how many cubes they will need, then test their prediction. Ask: Why were four cubes not enough? Why did you need eight times as many wooden cubes?

Chair Factory

Have students say how doubling the linear dimension changes the volume. For example, say: A factory manager had an order for a chair twice as big as this classroom chair. What measurements would you have to make? Ask students to make a three dimensional model of a chair using no more than about five cubes. Then have them double all the linear dimensions. Ask: How has the volume changed? Is the amount of change the same for different model chairs?

Sample Learning Activities

Grades 5–8: ★★ Important Focus

Enlarging a Picture

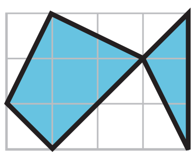
Photocopy a cartoon drawing onto square graph paper and have students enlarge it using a grid. Ask: If you double the length of each side, what happens to the area? What happens to the area when you triple the length of the sides? Can you predict what the area would be if you enlarged the dimensions four times? What changes (stays the same) in the enlargements? What is it about the shape that stays the same in each enlargement? Draw out that the lengths and areas change in a predictable way, but the size of the angles always stay the same.

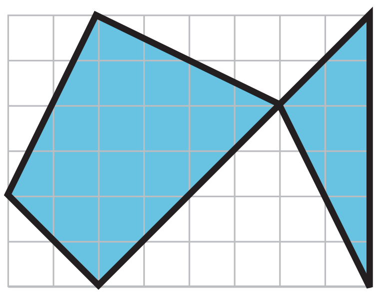
Reducing a Picture

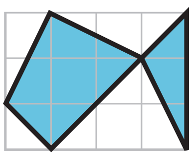
Extend *Enlarging a Picture* above by giving students a different, large picture on graph paper and asking them to reduce it. Ask: When you halve the length of each side, what happens to the area? What do you think will happen to the area if you reduce the dimensions to one third of their original size? What about the angles? Encourage students to test their predictions.

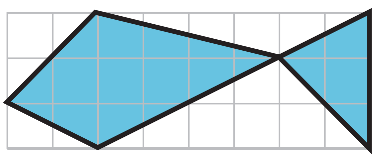
Changing Shape

Have students say what happens to the angles, lengths, and areas of shapes when the two linear dimensions are changed by different amounts. For example, invite students to draw a straight-line sketch of a fish on graph paper. Then, ask them to make the fish twice as long and twice as high. Invite them to measure the lengths of the two fish. Ask: How do matching lengths compare? Then, ask them to work out the area of the body of each fish. Ask: How do they compare? Have them measure the angles. Ask: How do they compare? Have students make another fish the same height as the original, but twice as long. Ask: How is the last fish the same as (different from) the original fish? Invite students to compare the matching lengths of the fish. Ask: How do they compare? What happens to the area? How have the angles changed? What differences do changes to angles make to shapes?









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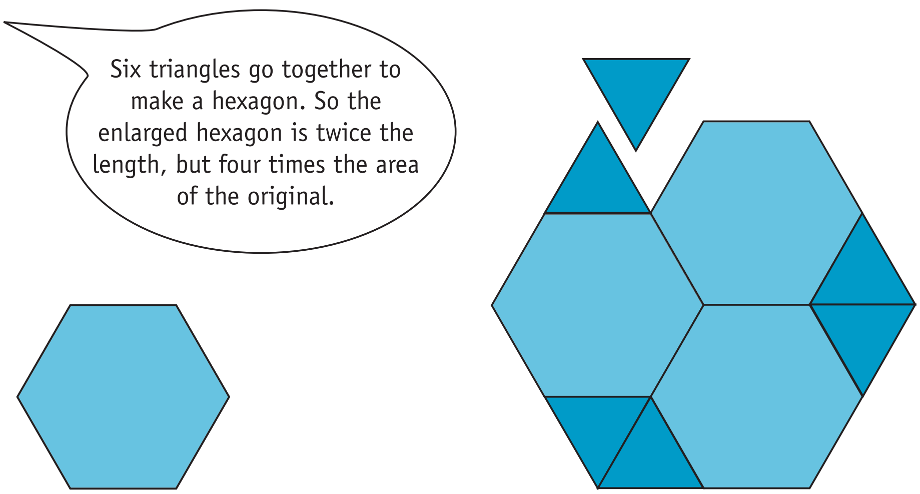
Grades 5–8: ★★ Important Focus

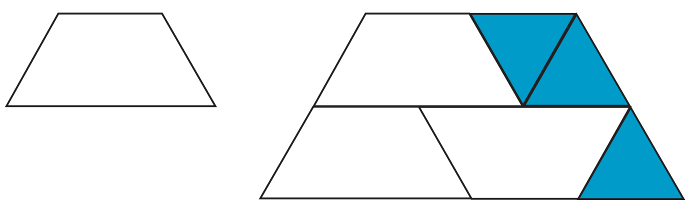
Flags

Have students each make a transparency of a simple geometric flag design. Use an overhead projector to project their designs onto large sheets of paper and, in some cases, turn the projector so that it is at an angle to the wall, distorting the image. Invite students to trace around their images so that each student has their original design and its enlargement (or, in some cases, its distorted enlargement). Have students compare the original transparencies with the copies. Ask: What is the same? What is different? Have you considered angles, lengths, and area? (See Case Study 2, page 54.)

Hexagons and Trapezoids

Extend the *Triangles and Other Shapes*, page 48, by having students use pattern blocks to make enlarged copies of the hexagon and trapezoid, doubling and tripling the dimensions. (You will need to use triangles to complete the enlarged shapes.) Ask: If you make a larger hexagon by doubling the dimensions, how many of the small hexagons would match the area of the larger hexagon? What if you triple the lengths of the sides? Can you predict how many of the small hexagons will match the area of the tripled hexagon? How does this fit with what you know about doubling and tripling the sides of other shapes? Will this work with trapezoids?





Using a Photocopier

Have students draw a curved, closed shape on a sheet of paper (like a curvy puddle), then enlarge (reduce) it for them on a photocopier using an enlargement (reduction) ratio of their choice. Invite them to compare their original with the copy. When comparing matching lengths, ask: How does it relate to the enlargement (reduction) ratio? Have students use graph paper to work out the area of each shape. Ask: How does the change in area relate to the enlargement (reduction) ratio?

Chair Factory

Extend *Chair Factory*, page 50, by tripling the linear dimensions. Record the number of pieces in the original chair and the enlarged chair. Produce a table from the results of the entire class. Ask: Is there a consistent relationship between the number of blocks in the original chair and the number in the tripled chair?

CASE STUDY 2

Sample Learning Activity: Grades 5–8—Flags, page 52

Key Understanding 2:
When two objects have the same shape:
- matching angles are equal
- matching lengths are proportional
- matching areas are related in a predictable way
- matching volumes are related in a predictable way

Working Towards: Measuring Phase and Relating Phase

TEACHER'S PURPOSE

My Grade 7 class had been enlarging and reducing figures using grids and had become quite skilled at it. They knew that on their grid enlargements each line increased in the same ratio and the angles did not change. Using grids meant, however, that the enlargement factors were always numbers like 2 or 3 or 1/2—a highly structured situation. I was not convinced that students really understood what it meant to be the same shape mathematically and how this related to enlargements and reductions.

MOTIVATION

Several days earlier, students had produced a simple flag design. I had given each of them a half-sheet of acetate to draw it on so they could show it to the class enlarged using the overhead projector. The day before my planned lesson, I turned on the overhead projector, and one at a time throughout the day, students put their flag on the projector, then traced the image produced on the wall on sheets of newsprint, which I had pinned up. The idea was that this would act as the template for making an actual flag in their art and technology lessons. During the day, I casually moved the projector so that the size of the images varied, but sometimes I pulled it around so that it was not square with the wall and so produced a distorted image. My students sit in groups, and I made sure that there was a mix of proper enlargements and distortions for each group of students.

My intention here was to provide some conflict for students as the distorted images would not have matching angles equal and matching lengths proportional. Distorted images are not the same shape.

CONNECTION AND CHALLENGE

The next day, I asked the students to get out the small (acetate) and large (newsprint) copies of their flags and asked the apparently simple question, "What stays the same and what changes?"

Students volunteered such things as, "They look the same, but different sizes" and, "The sides all go up the same amount."

I asked the students to think about the enlargements they had done with grids and the sorts of things they had investigated there. The students suggested angle, length, number of squares, area. I wrote these on the board. I then challenged the students to systematically investigate the relationship between the small and large versions of their flag so that they could report to each other.

Is there a relationship between matching angles on your two versions? If there is, what is it?

Is there a relationship between matching lengths on your two versions? If there is, what is it?

Is there a relationship between matching areas on your two versions? If there is, what is it?

ACTION

My students were used to such activities and quickly started working. Because the original was on a transparency, they compared matching angles easily by superimposing.

"Be very accurate," I said occasionally. "Are you sure?" I asked, when students started immediately to say that the angles did not change.

Within a few minutes, one or two students who had the distorted versions started to get uneasy, although a couple of others did not notice—either because they checked too few angles or because they were a bit casual with their superimposition and their distortions were not too great. I spoke quietly to some who were concerned and suggested they mark the angles that were equal and those that were not.

Although they had not finished with lengths and area, I drew the class together to talk about angle. "What did you find?" she asked.

Immediately students started to volunteer that the angles were the same. There was a chorus of agreement. I turned to one of the students who I knew had a distortion, "Yolanta, do you agree?"

"Sometimes," Yolanta said.

When I asked her what she meant, she held up her newsprint, pointed in turn at several angles and said, "These were close but these were bigger."

"Well," I challenged the class, "Many of you seem confident that the angles must stay the same, but Yolanta says hers were not. Who is right?"

A number of students then suggested that Yolanta's copying or measuring may have been incorrect, to which Yolanta objected. I asked Yolanta to hold up her original and her enlargement and as she did so Yolanta commented that it did not look right, but she had copied it properly. I asked whether any one else had found what Yolanta found and whether their copies also looked odd. A number of other students offered their own examples.

I then relented and asked Yolanta to bring her transparency to the front. She put it on the overhead projector and I asked the class to watch carefully. I began with the projector correctly positioned in front of the wall and then gradually moved it at an angle. The image on the wall changed shape; that is, it became more and more distorted. As the students began to realize what had happened, I admitted that I had set them up by moving the overhead projector the day before.

DRAWING OUT THE MATHEMATICAL IDEA

I spent some time drawing out from the students that the point of an overhead projector was that what was on the screen should look the same, only bigger, and that things look the same when they have the same shape. If they do not have the same shape, they look distorted (odd, lopsided, skewed).

I asked the students to look at their group's enlarged flags and decide which ones looked right (the same shape) and which looked distorted. The class then went around the groups one at a time and found that, as a rule, it was the people who had distorted copies who had found that their angles did not quite match. The class concluded the following together:

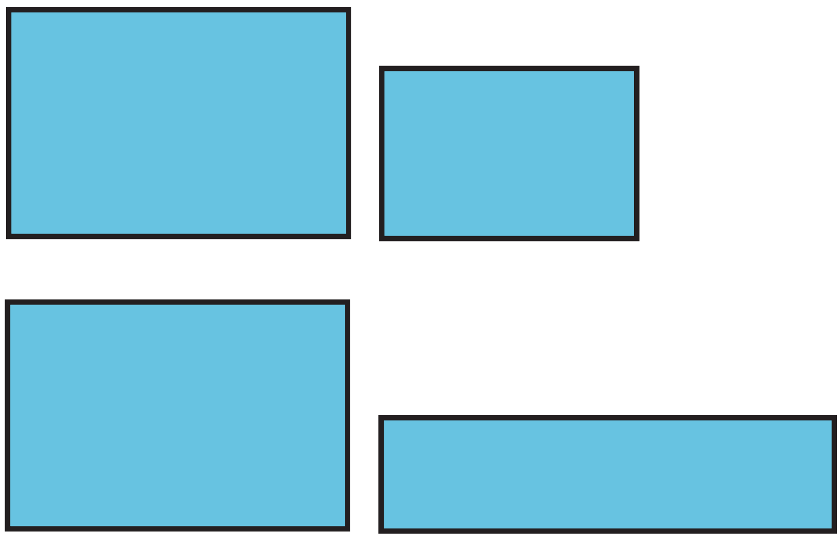
> If two figures have the same shape, matching angles will be the same.
>
> If matching angles are not the same, the figures are not the same shape.

CONNECTION AND CHALLENGE

I asked students to return to investigating the original question about length but taking into account which newsprint flags were enlargements and which were distortions.

One of the difficulties for students here is that they will often have formed the mistaken idea that all figures in a given class have the same shape; for example, they may think that all rectangles are the same shape.

This casual use of the term same shape can be quite confusing. We often refer to all rectangles as the same shape, but really mean they are in the same class of shapes. The first two of these rectangles are the same shape but the second two are not.



I suggested to students that they work out the ratios between matching lengths for six or seven different parts of the flag as accurately as they possibly could. They were then to decide whether they were all the same and whether there was a difference between the enlargements and distortions.

Most students were fairly confident that all the lengths would increase by the same amount, but this did not always translate into knowing that they needed to compute a ratio—that is, to divide. Measuring, deciding what computation to do, and computing accurately in order to find the ratio between matching lengths was a challenge for many students. This became the topic of the next few lessons.

For figures to be the same shape, the following must both be true:

- ***matching angles must be equal***
- ***matching lengths must be proportional.***

Key Understanding 3

Scale drawings and models have the same shape as the original object. This can be useful for comparing and calculating dimensions and for making judgements about position.

As suggested for Key Understanding 2, when two figures or objects have the same shape, we can predict the relationship between matching lengths, areas, and volumes on the two shapes. It is these mathematical relationships that make scale drawings, plans, maps, and models useful. A scale model of the three dimensional object will look like the original, only smaller or bigger, and similarly, a scale drawing of a two dimensional figure will look like the original. Often scale drawings give us two dimensional snapshots of three dimensional things, however. For example, maps and plans give a bird's eye view and only represent some features of the real thing. The focus of this Key Understanding is on the measurement involved in interpreting, using, and making scale drawings and models.

In the early years, there is little distinction between this Key Understanding and Key Understanding 2. Students should build on their intuitive ideas about scale with the emphasis being on what looks right but is bigger or smaller. For example, students may match component parts according to a rough scale (e.g., *This chair is for the baby bear*) and attempt to make models of familiar things, discussing how they could make it look right (e.g., *It does not look right because the wheels are too small for the car*). Later, they should build scenes (dioramas, model farms, dollhouses) to an intuitive scale, also asking whether it looks right and, if not, deciding how to improve it.

During the middle and later years, there should be a gradual development from a very intuitive feeling for scale to the somewhat more formal use with whole numbers or unit fractions as scale factors. There are two aspects to this Key Understanding. First, students should learn to interpret and use the information provided by scale drawings, plans, maps, and models to make decisions such as whether the house will fit on the block, how far it is between the two towns, what the shortest route is and if, on average, an adult is 1.8 m tall, about how tall that building is likely to be.

Second, students should produce scale drawings, plans, maps and models in order to provide information to others and to make decisions about such things as the arrangement of furniture in their classroom, stage props for their play, or how a flag design they have in mind will look when made. They should attempt to make accurate scale drawings of simple figures and objects, such as a plan of the school to provide to visitors, or a storage box. To produce a scale drawing or model, they will need to decide what measurements to take on the original. This may require them not only to consider lengths but also angles. It is important for students to have the opportunity to make such decisions for themselves so that they learn what can go wrong when they take insufficient or unhelpful measurements.

Links to the Phases

Phase	Students who are through this phase...
Quantifying	■ attend to scale informally when interpreting and producing maps, plans, drawings, or models ■ realize that they can get a sense of comparative distances and lengths from maps, plans, and models produced by others ■ attempt to adjust sizes to get their own maps, plans, drawings, or models to look to scale
Measuring	■ use simple scale factors to calculate and estimate measurements *For example*: Given a picture or object and asked to make it three times as big or one third as big, a student will work out the size of the parts in the scaled version. ■ are able to work out what measurements to take when making straightforward scale drawings, maps, or plans
Relating	■ are able to compute the scale factor between two different-sized versions of a figure or object ■ are able to use data in a map, plan, or photograph together with their everyday knowledge to estimate scale factors and use the data to answer other questions about the objects represented

Sample Learning Activities

K–Grade 3: ★ Introduction, Consolidation, or Extension

Model Farm
Give students a three dimensional model of a town or farm that has some parts of it to scale and others not. For example, in a model farm, include a plastic duck that is as big as the shed, and a model tractor that is as small as the dog. Alternatively, ask students to construct their own models. Ask students to identify what looks right and what does not. Ask: Why do some things look the right size and others do not? How could we fix those parts that do not look right? Encourage comparative language (e.g., *the tractor has to be bigger than the dog because real tractors are much bigger than real dogs*).

Teddy Bear
When students are illustrating stories such as The Three Bears, give them a teddy bear to copy. Before drawing, focus the students on the shapes they can see in the ears, face, body, and legs. Ask students to draw Father Bear first, then to redraw it smaller to be Mother Bear and smaller again to be Baby Bear. Ask: What did you need to think about when you made Mother Bear's and Baby Bear's heads? What about the ears? How are they different on Baby Bear? What is the same about the ears in all three drawings?

Different-Sized Dolls
Show two small dolls or action figures and ask students to find various items that are the right size for each model to use. For example, ask: Why would this marker cap make a good glass for this doll? Why do you think it would be too small for that action figure? Would the thimble make a good wastepaper basket? Why? Why not?

Classroom Plan
Have students draw bird's eye view plans of the classroom to show different possible arrangements of desks. Ask them to first build a three dimensional model using blocks or building bricks for furniture, then develop a two dimensional plan from their model. Ask: Would there be enough room to walk between your desks? Should the desks be larger or smaller than the computer bench? Why is there not enough room on your plan for all the desks in the room? What could you do to fix it? (See Case Study 3, page 66.)

Sample Learning Activities

Grades 3–5: ★★ Important Focus

Making Models

Give students materials to make models of familiar structures (towers, robots, houses, bridges). Ask them to think about the relative size of parts of their models. For example, prompt their thinking by asking: If the house is this big, how big do you think the window should be? How big would the door be? Does it look right? Why? Why not? What would you do to fix it?

Spiders

Ask students to imagine they are a spider on the ceiling of their classroom looking down. Ask: What do you think the top of the desks would look like? What about the bookcase and the cupboards? Do you think you could see the legs of the desks? Have them draw a plan of the classroom as they think the spider would see it. Ask: What looks right on your plan? Does anything look strange? (*My plan looks like the classroom but it is a bit funny because I have drawn the teacher's desk too big for the other furniture*.)

Informal Scale Models

Have students make models to a specified but informal scale. For example, give students a box and say: This is a table. Make a chair the right size for this table. Ask: Why did you decide to make the legs that high? What other measurements did you have to think about to make the chair the right size?

Scale Factors

Have students investigate simple scale factors on maps (e.g., 1 cm = 1 km). Ask: What does this mean? How can I use this to work out how far it is from Melanie's house to the video store? Invite students to work out approximate lengths and distances using simple scales.

Scale Models

Have students use uniform units (e.g., straws, craft sticks) to measure given features of the environment (e.g., the width, length, and height of furniture or playground equipment). Ask them to make models of the measured features, using 1-cm or 2-cm (or Base Ten) cubes to represent one unit. Encourage students to explain the scale they have used. Ask: How do you know that the table should be three blocks tall? How many straws long was the playground tunnel? So, how many blocks long will it be in your model?

Grades 3–5: ★★ Important Focus

More Classroom Planning

As a whole class, have students help measure the length and width of the classroom and the fixed furniture around the room. Draw a one-tenth scale plan (1 dm = 1 m) of the classroom, including only the fixed furniture, on a large sheet of paper. Help students measure and make correctly scaled cardboard cut outs of their desks and chairs, as well as other moveable furniture. Invite them to position their cut outs of the classroom plan to show how the furniture is currently arranged. Later, have students re-position the furniture on the plan to help work out a new arrangement of furniture for the classroom. Ask: Have you left enough space between your desks? How do you know? Will there be enough room to walk between those chairs?

On the Computer

Have students investigate reducing and enlarging print on the computer in order to predict and check scale changes. Ask them to type the same word a number of times using the same font. Invite them to change each one to a different point size, record the size of each one next to it, and print the sheet. Have students hand their sheet to a partner and ask, *How much bigger or smaller have I made my word*? Encourage students to predict whether it is, for example, twice as big, three times as big, four times as big, half as big. Ask: How do you know? What effect does the different point size have on the size of the print? Is a 40-point word twice the height (width) of a 20-point word?

kitten 10 pt

kitten 20 pt

kitten 40 pt

Sample Learning Activities

Grades 5–8: ★★ Important Focus

Floor Plans

Have students examine the floor plans of houses and identify some rooms to work out the scale used. Invite them to use the sizes indicated on the plan for each room to measure out the actual dimensions on the playground. Ask: How many square metres is the games room in real life? What are the measurements on the plan? How can you work out what scale the architect has used to draw the plan?

Distorted Simple Shapes

Ask students to examine two drawings made up of simple shapes, one supposedly an enlargement of the other but with some distortions so that it does not look right. Ask: Which parts of the enlargement look right? Which parts do you think are distorted in some way? How can you tell? Invite them to use grid lines to discover exactly what is wrong and then attempt to correct it. Ask: What do you need to do to correct the roof? How have the windows changed?

Athletics Banner

Invite students to design a banner using a scale drawing. For example, say: The Canadian Athletics Team needs a new banner to take to the World Athletics Championships. The banner has to be 4 m by 2 m and include the words "Canadian Athletics Team." Make a scale drawing of it so the manufacturer knows exactly how to make the full-sized banner. Ask: Why is it important to tell the manufacturer the scale used to design the banner?

Grades 5–8: ★★ Important Focus

Dolls and Action Figures

Ask students to bring in dolls and action figures. Invite them to compare the height of the doll to their own height and find how many times the doll's height would fit into their own. Have them use this scale factor to compare other body measurements (e.g., waist, length of limbs). Ask: If your height is ten times the height of the doll, what would you expect the doll's waist measurement would be? How can you check? What other measurements are different from what you would expect? Why do you think dolls are not accurate scale models of real people? What about toy animals? Are they also distorted models of real animals? How? Why?

Micronians and Earthlings

Say: The inhabitants of the planet Micros look exactly the same as humans, but their forearm bone is only 10 cm long. Ask: Can you use this information to work out how tall they are? Ask students to draw a picture to scale that shows an Earthling standing next to a Micronian. Ask: What other measurements do you need to make to complete the drawing? How can you use the information you have about the Micronian's forearm to decide how long its legs are? Help students see that by dividing the length of their own forearm by 10 cm (the length of the Micronian's forearm), they will arrive at a scale factor that tells them how many times longer their legs (waist, chest, hands) are than the Micronian's.

Cereal Boxes

Have students compare the picture on the front of small and large cereal boxes. Ask: What has remained the same? What has changed? Invite students to draw a grid across the front of the smaller box and create a smaller version of the same box. Ask: What scale factor have you used?

Scale Drawing of the School

Give students a scale drawing of the school and ask them to work out the scale factor used. Invite them to measure different parts of the school and compare their measurements to measurements taken directly from the drawing. Ask: How have you compared the two measurements? How can your calculator help you compare? Which operation did you use to work out the scale factor? How can you check that you have the correct scale factor for the drawing?

Display of Artwork

Have students use scale drawings to plan arrangements of objects. For example, say: We need to help the librarian set up a display of winning artwork from the artwork competition. Two of the winning pieces are 30 cm by 21 cm and the other two are 60 cm by 42 cm. The bulletin board is 1.5 m by 2 m. Encourage students to choose a simple scale (e.g., 1 mm = 1 cm) to draw the bulletin board and make cut outs of the winning pieces to experiment with arrangements. Then, invite students to choose a suitable arrangement and use the scale plan to set up the full scale display in the library. Ask: What is the same and what is different in the scale plan? What do you need to measure to be sure the winning pieces are set up as planned?

CASE STUDY 3

Sample Learning Activity: K–Grade 3—Classroom Plan, page 60

Key Understanding 3: Scale drawings and models have the same shape as the original object. This can be useful for comparing and calculating dimensions and for making judgements about position.

Working Towards: Quantifying Phase

MOTIVATION AND PURPOSE

My Grade 3 students had talked about how they could rearrange the classroom and I decided to use this to develop ideas about scale. I asked the students to make a desktop model of how they would like the classroom to look.

ACTION

A few students chose wooden cubes to represent desks, but most decided that Lego™ pieces were better. Several students chose a sheet of paper to use as the floor and some drew in furniture around the room before counting the correct number of Lego™ desks. I showed the rest of the class what they had done and suggested that they could all draw a plan of their classroom layouts.

Students chose the size of paper they wanted to use and many started by using the Lego™ pieces as templates to draw around so that all of the desks would be the same size. Others put the Lego™ pieces away before they began their plan and seemed to not consider the size of the pieces of furniture as they drew them in. Quite a few students ran out of floor space before they had drawn in all the desks, and decided they would have to start over with a larger piece of paper. Re-drawing smaller desks was not the obvious solution for them. Others had drawn very small desks and so ended up with lots of floor space in their plans. None of the students used any form of measurement in drawing up their plans.

I brought the students together and asked, "Is there enough space on your plan for people to walk around the room between the desks? Everyone in the room has to be able to get to the door easily."

I wanted the students to think about the size of the various pieces of furniture in relation to each other and to the size of the room.

I asked students to use their plan and show their partners how people would walk to the door. Many students realized they had not considered the space between the various pieces of furniture. Some realized that they had made some pieces bigger or smaller than they should have and made comments like, "But the teacher's desk is not that big really, it is just a bit bigger than our desks. All of your desks are kind of squashed up in the corner and there would not be enough space for chairs. I could not fit through that gap, it is too small."

DRAWING OUT THE MATHEMATICAL IDEA

After a while, I called them together again and said, "When we draw a plan, we really need to try to have things about the right size."

I drew a rectangle on the white board and asked, "If this is the size of one desk in our room, how big should the computer table be? Bigger than it or smaller than it? How much bigger than it?"

I asked everyone to show me the size with their hands and then asked Tadao to draw it. The class discussed whether this was about the right size and after a small correction I asked, "If this is the size of one desk in our room, then how big should my desk be?" Again, the students indicated with their hands, and one student drew the picture.

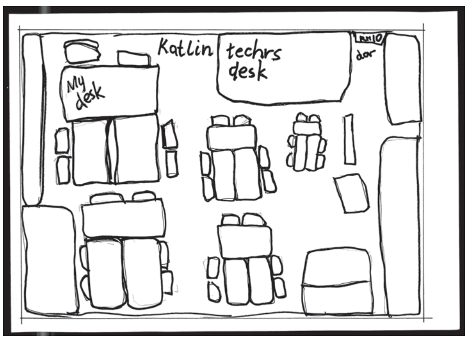
"OK, now look at your plan. Is the computer table about the right size in comparison to one desk that you have drawn?" I asked. At this point, many of the students seemed concerned that their plan was not right so I offered them the opportunity to try again.

It was time to draw out the importance of trying to keep things to scale.

Several students did not attempt the conventional bird's eye view. They drew the desks with the legs showing and included the wall displays and chalk-board, as well as students sitting in their chairs and the teacher walking around.

Corey's plan of the classroom

Katlin's plan is typical of the plans initially produced by most other students. They used a conventional bird's eye view, but most did not attend to the relative size of furniture or space between the desks.



Katlin's plan of the classroom

Key Understanding 4

We can calculate one measurement from others using relationships between quantities.

In the everyday world, many of the measurements we use have not been obtained directly but have been derived from other measurements by undertaking computations. This may involve the following.

Choosing and using an operation, such as:

- adding the quantities shown on each of the containers to decide how much ice cream there is in the freezer
- weighing ourselves on the bathroom scales, weighing ourselves holding the cat, and find the difference to find the weight of the cat
- measuring the thickness of a thousand sheets of paper and dividing the measurement by one thousand to measure the thickness of a sheet of paper

Choosing and using a rate or scale, such as:

- finding the volume of a container by finding the mass of the water it holds, and using the fact that water weighs one gram per cubic centimetre
- estimating the time it will take to travel between two towns using the anticipated speed (a rate) and the distance
- using a measurement on the map, and the scale factor of 1000 to estimate a real distance

Choosing and using a formula, such as:

- finding the area of a rectangle by measuring the lengths of two adjacent sides and multiplying the two measurements
- using a baby's weight and a formula relating the amount of medicine needed to body weight to work out the right dose of medicine

Students should learn to recognize when a computation will help solve a practical measurement problem, work out which computations to do and do them correctly.

Working out whether and when a computation is possible involves thoughtfulness and judgement. For example, students may have learned through activities such as those described in Key Understanding 1 that the area of a rectangle can be found by multiplying its length by its width. Confronted with the problem of finding the area of a garden, they then have to decide whether they can use this rule or formula. *Is the garden a rectangle? Can we check? Is it close enough for my purposes? If not, can I break the garden up into smaller rectangles that I can find the area of?* and so on.

If the students decide that a particular formula may be used, they will need to decide what component measurements are required and apply the formula correctly. Applying the formula correctly is not simply a matter of computational skill, it involves first checking that the units of measurement are appropriate and doing any needed conversions.

Links to the Phases

Phase	Students who are through this phase...
Quantifying	■ are able to choose operations in relatively straightforward situations *For example*: Students may add the lengths of the sides of a shape to find its perimeter, or subtract a television program starting time from its finishing time to work out if the three-hour videotape is long enough.
Measuring	■ are able to carry out computations with measurements involving decimals ■ use the relationship between quantities to work out one quantity from another and will make some of their own measurement short cuts *For example*: Students might multiply the length of one side of a regular polygon by five to get the perimeter or find the volume of a prism composed of cubes by multiplying the number of layers by the number in each layer.
Relating	■ are able to choose and use straightforward formulas with which they are familiar, including working out what measurements they need to make in order to use the formula and ensuring that the units are consistent

Sample Learning Activities

K–Grade 3: ★ Introduction, Consolidation, or Extension

Incidental

Let students see and hear your computations when you combine measurements for a purpose. For example, while planning the assembly, say: We will allow about two minutes for the speech by Ms James and three minutes each for the two songs, so that is eight minutes so far. Or, say: We are going to need two cups of starch for each batch and there will be three batches, so we will need six cups of starch.

Does It Work?

After students have used a common unit to measure the length of various paper tapes, or ribbons, ask them to predict the total length if the ribbons were to be arranged in a long line. Ask: What would we need to do to work out the total length? Would your calculator be helpful? Have them check their computations by laying out tapes (ribbons) end-to-end and measuring the total. Repeat for other combinations of lengths.

Class Party

Invite students to solve problems that involve combining quantities. For example, say: In preparing for the class party, Mrs. Williams poured one cup of powdered drink mix into the jug and then added nine cups of water. Ask: How much drink did she make? How do you know it is that much? What if she wanted to make double that amount? How many cups of mix and how many cups of water would she need? How did you work it out?

Cooking

In cooking activities, involve students in planning the quantities and writing out new recipes (doubling the ingredients for a cake, making enough dough for two cookies for each student, computing the amount of ingredients for home-made lemonade from a recipe for four).

Sample Learning Activities

Grades 3–5: ★★ Important Focus

Excursions

Have students help plan excursions. For example, invite them to work out when they will return to school. Ask: How long will we spend at the destination? How long will it take to get there? How long will it take to get home? How long will we spend having lunch or snacks? Help students see how periods of time are combined and related to the starting times to enable them to tell parents when they will return to school.

Borders

Have students work out the length of a border for the bulletin board. Encourage them to decide on a suitable unit, count how many on each side and attach that number to each side. Ask: What do you notice about the top and bottom and side measurements? Invite them to work out the total length without re-measuring. Ask: Can you see a short cut for measuring another bulletin board?

Frame a Picture

Invite students to work out how much cardboard they need to frame their art. Ask: What measurements will be needed? How will the corners go together? Will this make a difference to the measurements? How can we work out how much cardboard will be needed altogether? The cardboard comes in lengths of 1 m, 1.5 m, and 2 m. Which would be the best to use? Have students measure and construct their frames.

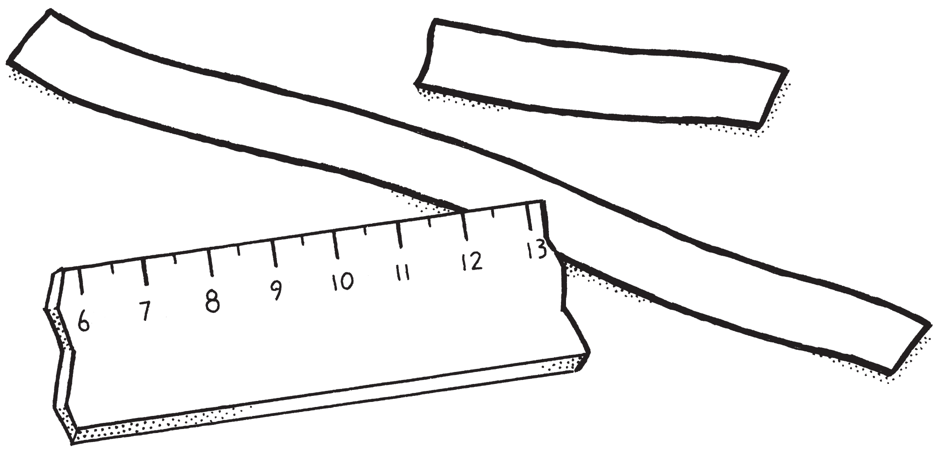
Overcoming Limitations

Have students overcome limitations in the measurement range of equipment. For example, give students kitchen scales that weigh up to 250 g. Ask them to find the mass of a bag of flour (which weighs more than 250 g). Then, ask them to find about 400 g of tomatoes to use in a sauce. Ask: What computations did you need to do? How can you check your results a different way?

Grades 3–5: ★★ Important Focus

Broken Ruler

Ask students to measure and compute to overcome inaccuracies in equipment. For example, give students parts of a broken ruler (they should have different parts) or paper tape marked like a broken ruler. Ask them to find the length of both shorter and longer items and say how they were able to measure in centimetres. Ask: How can you work out the length without having to count each centimetre gap? What calculations can you use? Which measurements involved more calculations? Why? Does it matter which part of the ruler you have? Should you get the same result?



Combined Mass

Ask students to find objects that have a combined mass of 1 kg (combined length of 1 m). Ask: How did you do this? Did you need to use any computations? How did you find out how much the last object had to weigh (measure)? Was it difficult to find something that was just right?

Weighing Awkward Objects

Have students use bathroom scales to find the weight of objects that can be difficult to weigh without special scales (e.g., small animals, bags of fruit or vegetables, a packed suitcase) by weighing themselves, weighing themselves together with the object, then computing the difference. Ask: Why does this method work? How accurate is it likely to be? Could we weigh a very small kitten using this method? Why? Why not?

Sample Learning Activities

Grades 5–8: ★★★ Major Focus

Recycled Cans

Have students investigate the amount of refund the school receives for recycled aluminum cans. Invite them to use this information to calculate the total mass of the cans collected by their class so far, the total mass of cans collected in the school, the amount of money the school will receive, and a prediction of how much money the school will receive by the end of the year. Ask: How can you work out the total mass when you cannot fit all of the cans on the scales? What do you need to know to work out how much money the school should receive?

Overcoming Limitations

Invite students to find a way to measure things that are too small for the accuracy of the equipment available (the thickness of a single piece of paper using only their ruler, the mass of a grain of rice using kitchen scales, the volume of a drop of water using a measuring cylinder). Compare the methods and the operations used for each measure. Ask: Which were the quickest and easiest to carry out? Did the different methods produce different answers? Why did this happen? How could the range be reduced?

It Needs Fixing

Ask students to compute to address inaccuracies in equipment. For example, say: Our tape measure has stretched, so when I use it to measure an object that my stretched tape shows is 1 m long, I know that the real length of the object is 1.2 cm longer than 1 m. Ask:

- What would the real length of the room be if my stretched tape measure shows it as 4 m long?
- What would be the real length of a chair that my tape measure shows it as 50 cm?
- What would be the real length of my desk that my tape measure shows it as 1.5 m?

Have students share the computations they used to work out the real lengths.

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Grades 5–8: ★★★ Major Focus

Dripping Tap

Have students measure the quantity of water wasted from a dripping tap in one day. Ask: Is there any way we could work it out without leaving the bucket under the tap all day? How could we use this information to work out how much water would be wasted in a week? Invite students to work it out. Then, ask: Which measurements did you need to make? What computations did you need to do? How could you adjust the time measurement to make the computations easier?

Oil Spills

Have students find the areas of a range of irregular regions (not given on grid paper), such as the aerial view of oil spills. (See *Oil Spills*, page 32.) Limit the materials students can use to paper tiles, ruler, pencil, and calculator. Invite students to explain how they worked it out. Ask: How can you use the length-by-width rule to avoid counting all the squares? (See Case Study 4, page 78.)

Using Perimeter

Extend *Oil Spills* above by having students test the incorrect hypothesis that you can use the perimeter of a region to work out the area. For example, present the following conflict situation. Say: Someone in the other class found a very quick and easy method to work out the area of a diagram of an oil slick. They taped string around the edge of the shape then cut and joined the ends of the string. They then made the string into a rectangle, and multiplied the height and the width measures of the rectangle to work out the area. Ask: Do you think this method would give you a measure of the area? How do you know? How could you test this? How would you convince the student from the other class?

Using a Formula

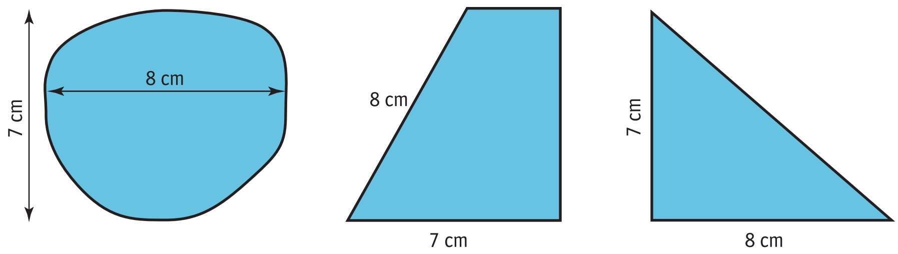
Have students decide when it would make sense to use a particular computation or formula and when it would not. For example, present the following problems and ask: Would it make sense to multiply 4 by 10 to get an answer? Why? Why not?

- A man can run 1 km in 4 min. How long would it take him to run 10 km?
- A kilogram of apples costs $4. How much would it cost for 10 kg?

Present the following questions and ask: Could you sensibly use the length-by-width rule to answer the following questions? Why? Why not?

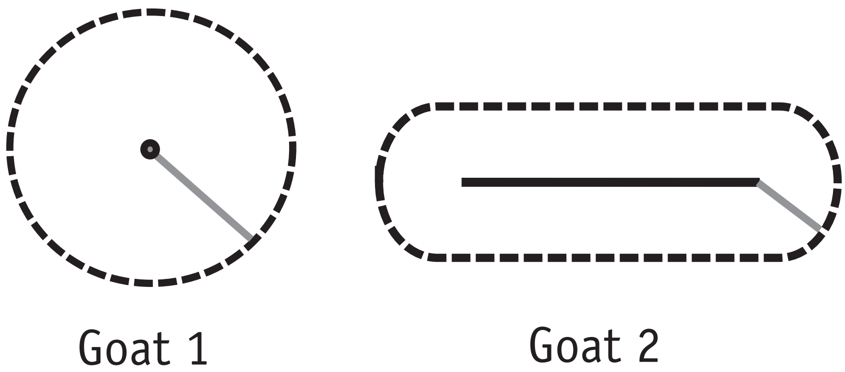
- The school athletic field measures 70 m by 50 m. What is its area?
- A rectangular pasture measures 70 m long and 50 m wide. What is its area?
- A rectangular park is 70 m long and 50 m wide. How much fencing will be needed to enclose it?

Show students the following shapes and ask: Why would it not make sense to multiply 7 times 8 to find the area of these shapes?



Grazing Areas

Say: There are two goats. The first goat is tethered by a rope to a stake in the ground. The second goat is tethered by a rope half as long as the first goat's rope to a sliding rail that is double the length of the first goat's rope. Invite students to use a compass and ruler to draw representations of the two feed areas. Give students cubes, paper tiles, or pencil and ruler to work it out and then ask: How did you work out which animal has the larger grazing area? Encourage students to discuss and justify the method they chose. Draw out the strategies that were the quickest and easiest to use. Ask: How do you know these strategies are as accurate as counting all the squares?



Grades 5–8: ★★★ Major Focus

Area Problems

Have students solve practical area problems involving regular and irregular regions (how much fertilizer needs to be purchased for the school athletic field, how much paint needs to be ordered to paint a large red circle on the asphalt using two coats). Encourage them to draw on a range of strategies, including partitioning into rectangles and other regions and adding areas, using formula, and so on. Ask them to explain and justify their strategies to their peers.

Postal Rates

Give students information about postal rates from Canada Post. Say: Franco wants to send some small gifts to Italy. They have a total mass of 1.4 kg and each weighs between 100 g and 150 g. Ask: What is the best way to send them? Invite students to investigate and compare sending them together in one parcel or separated into two or more parcels, by surface mail or airmail. Ask: How would your choices be different if the gifts had to get to their destination as soon as possible?

Missing Labels

Have students complete measurement problems by filling in the missing labels or measurement units for each answer and then justifying their choice. For example, ask: How do you know which unit is needed for these answers?

- What is the area of a 4 cm by 3 cm rectangle? 12 ___
- What is the volume of a 5 cm by 2 cm by 3 cm rectangular prism ? 30___
- How far will I travel if I drive at 95 km/h for $1\frac{1}{2}$ h? 142.5___
- The scale on the map is 1 cm = 5 km. If the distance between towns on the map is about $3\frac{1}{2}$ cm, what is the real distance between the towns? 17.5 ___

How Much Concrete

Have students solve area and volume problems in which different units are used. For example, say: Compute how much concrete in cubic metres is needed to make a path 50 cm wide, 20 m long, and 50 mm thick. Would 1 m^3 of concrete be enough or would you need $2m^3$? How did you decide? Encourage students to use diagrams and visualization. For example, invite students to imagine how many layers of path would reach a metre tall. Ask: How can you compare the thickness in millimetres to a metre?

Can You Do It?

Have students work in pairs or groups to decide the answer to questions like the following:

- If the area of the square of carpet is 49 m^2, can you work out the length of the sides?
- If the volume of a cube is 8 cm^3, can you work out its surface area?
- If the area of a rectangular swimming pool is 18 m^2, can you work out the lengths of the sides?
- If the volume of a rectangular prism is 24 cm^3, can you work out its surface area?
- The area of the pasture is 800 m^2. Can the farmer work out what fencing he needs?

Ask: How did you decide which measurements you could work out? What computations would you use? Why can you not work out the other measurements?

Did You Know?

Many students believe that you can work out the area of a shape from the perimeter. This is true for squares and for circles, but it is not generally true. It seems that students who have learned to think of area simply as length times width will try to use it even when it does not help. For example, asked to find the area of an oil spill, many students in Grades 5 to 7 placed a piece of string around the edge of the spill and then formed the string into a square or rectangle so they could use a formula to work out the area.

Students need many experiences that help them distinguish between the attributes of perimeter and area and realize that one figure can have a bigger perimeter than another but a smaller area and vice versa. For example, have students:

- produce different figures all with the same perimeter and then put them in order by area
- produce different figures all with the same area and then put them in order by perimeter
- arrange various figures in order first by area and then by perimeter and compare the orders

CASE STUDY 4

Sample Learning Activity: Grades 5–8—Oil Spills, page 74

Key Understanding 4: We can calculate one measurement from others using relationships between quantities.

Working Towards: Measuring Phase and Relating Phase

TEACHER'S PURPOSE

I was considering the pointers to achievement for Measuring phase and Relating phase that suggested that students should be able to dissect irregular shapes into rectangles in order to find the area. My Grade 6 students had measured things like leaves and puddles earlier in the year (developing their understanding of direct measure), but of late had mostly been using the length-by-width formula for rectangles. I wondered if they really understood what they were doing and why, and thought returning to irregular shapes might help them clarify when and how the formula could be used.

PURPOSE AND MOTIVATION

A recent oil tanker accident resulting in a large oil spill had provoked an animated class discussion, so I decided to use that context as a basis for the area activity. I gave the students a diagram of a large, irregularly curved, closed shape that could not easily be approximated by a rectangle. I told them the following scenario: "Here is an aerial view of an oil slick. The company needs a fairly accurate estimate of how much surface it covers to work out costs of treatment. What is the area of the oil slick?"

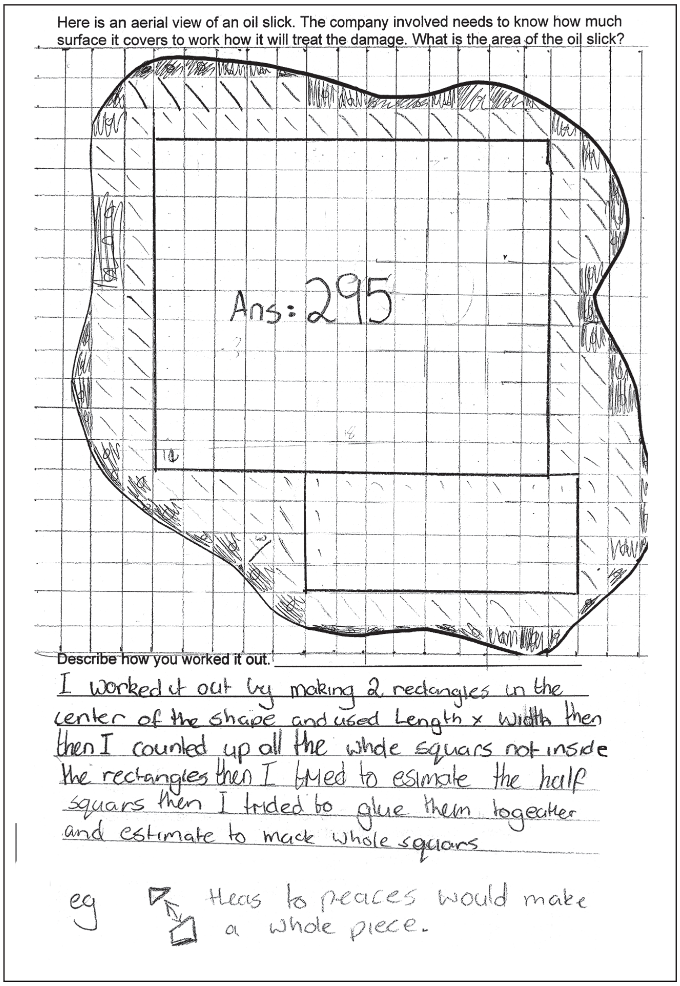
CONNECTION AND CHALLENGE

I provided Base Ten squares, but not square graph paper. I wanted to provoke the students to use a strategy other than counting squares. Many students started placing the squares on their shape, but quickly realized this was going to be too much of a chore, so they started drawing grid lines on their sheet. A few students started groaning about there being too many squares to count. At that point, I asked, "Can you find a shorter way to work it out rather than counting all the squares?"

ACTION

Many students still chose to count the squares. Some students, however, seemed to be excited by the challenge of finding a shortcut.

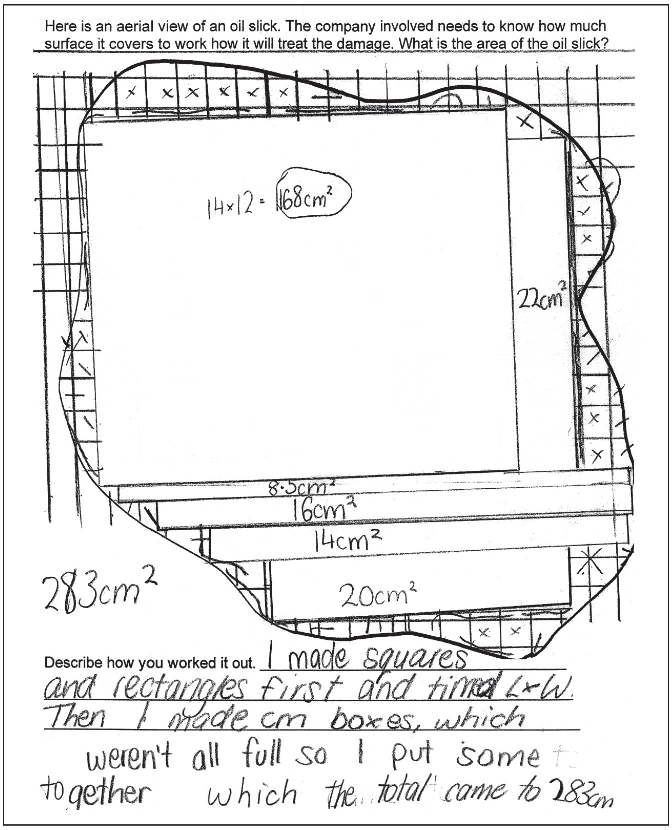
About eight students around the room used the idea of drawing rectangles inside the shape. Some students, like Megan, drew a large and a small rectangle over the grid she had drawn and then multiplied the length and width to work out the squares in the rectangle. She then counted the whole and part squares outside the rectangles.



Megan's work sample

I knew it was common for students to think that if shapes have the same perimeter they also have the same area, and to apply area formula to any shapes. I made a note of these misconceptions to be dealt with in the follow-up lesson.

Other students, like Jessie, had drawn rectangles inside the shape first, then constructed the square grid outside the rectangles. They calculated the rectangle areas, added them and then counted the leftover squares.



Jessie's work sample

Lori asked if she could use string. She carefully placed it around the edge of the shape, then formed the same length of string into a rectangle and used length by width to work out the area. Several others followed suit. John made his shape into a square and found the area of the square using the formula he knew.

Two students had come up with a formula for irregular shapes like the oil spill. Their formula was perimeter times pi. They wanted to know if this was correct.

At that stage I brought the class together. I began by asking students to call out their estimates and I quickly wrote them on the board. There was a fairly big range. I asked the students how they could explain this, and whether the range was acceptable. Some were happy with the range, but a few thought that some of the suggestions were "way off." After some discussion, I drew out from the students that some of the strategies we used might not have given as good an estimate as others.

This gave me the opportunity to focus on the strategies they had used. I started with the rectangle idea and suggested to the class that the other shortcut ideas would require another lesson to investigate.

I asked Megan to explain why she drew a rectangle on her grid. Megan showed the class her diagram.

"If I drew the rectangle I could multiply the number of squares in each row by the number of rows using the calculator. Then I found another rectangle underneath and I did the same thing. Then all I had to do was just count the leftover whole squares, and count up the parts of the squares around the edge. It was easy."

I added, "So, you found that looking for an array in your grid made it quick and easy to work out how many square centimetres in your shape. Who else looked for arrays?"

Students who used a similar method to Megan's showed their drawings. I focused their explanations on how they used the arrays. I asked Jessie to explain how he worked his out because he had also used rectangles but his method was different from Megan's.

"Well, I couldn't be bothered drawing in all the squares. I knew that if my rectangle was 14 cm across and 12 cm down that is the same as saying 12 rows of 14 which is 168 square centimetres. I did the same thing with the other rectangles."

Two others shared the way they worked out the area of the rectangle within their shape and how they dealt with the leftover squares.

DRAWING OUT THE MATHEMATICAL IDEA

I asked, "So what do you have to do to be able to use these shortcut methods?"

"You have to look for rectangles on your grid and then see how many rows and how many columns you have, then you can multiply," said Sharn.

Leia added, "You do not have to count all the squares in the shape and get mixed up if you do it like that."

"You could draw a big rectangle over the whole shape and then take away the squares that are not in the shape," said Nathan.

Some students thought that was a good idea and said that they would try that next time.

"It is just easier if you know the area of a rectangle is length times width," said James.

I knew that they would need to do many more of these types of activities so that students who were still counting each square could come to believe in the more efficient way of working out area over time.

I closed the lesson by saying there were two important jobs to be done over the next few days. The first job was to test the string shortcut method and the perimeter times pi method to see if they worked.

The second job was to examine the range of answers to see if the students could decide which answers they would accept as being in the correct range and to talk about how accurate they were and needed to be. In doing this, they would need to think about the scale of the drawing and what they would actually be able to tell the oil company.

Chapter 3

Estimate

Make sensible direct and indirect estimates of quantities and be alert to the reasonableness of measurements and results.

Overall Description

Students have a good feel for the size of units, make sensible estimates in commonly used standard units, and have the disposition and skills to judge the reasonableness of estimates and measurements. They know that to estimate which of two rocks has the bigger volume, looking may be sufficient, but to compare their masses, lifting the rocks will probably be necessary. They have a range of benchmarks that they use in estimation; for example, they may know which of their fingers is about a centimetre wide, what a litre container of milk looks like, and how heavy a kilogram of butter feels. They also use these benchmarks to judge the reasonableness of measurements and estimates, saying, for example, that the average height of students in their class cannot be 2.3 m—they must have made a mistake. Students also reason from familiar or known quantities to estimate quantities that cannot be found directly or conveniently; for example, yearly water wastage from the school's leaky taps or how many apples are eaten in their province or territory each day.

Indirect Measure:
Key Understandings Overview

Teachers will need to plan learning experiences that include and develop the following Key Understandings (KU). These Key Understandings underpin achievement of this family of concepts. The learning experiences should connect to students' current knowledge and understandings rather than to their grade level.

Key Understanding		Description
KU1	We can make judgements about order and size without actually measuring. We should think about how confident we can be of our estimate.	page 86
KU2	We can improve our estimates by getting to know the size of common units and by practising judging the size of objects and events.	page 94
KU3	We can use information we know to make and improve estimates. This also helps us to judge whether measurements and results are reasonable.	page 108

	Grade Levels—Degree of Emphasis			Sample Learning Activities
	K-3	3-5	5-8	
	★★	★★	★★	K-Grade 3, page 88 Grades 3-5, page 90 Grades 5-8, page 92
	★★	★★	★★★	K-Grade 3, page 96 Grades 3-5, page 99 Grades 5-8, page 102
	★	★★	★★★	K-Grade 3, page 110 Grades 3-5, page 111 Grades 5-8, page 113

Key

★★★ **Major Focus**
The development of this Key Understanding is a major focus of planned activities.

★★ **Important Focus**
The development of this Key Understanding is an important focus of planned activities.

★ **Introduction, Consolidation or Extension**
Some activities may be planned to introduce this Key Understanding, to consolidate it, or to extend its application. The idea may also arise incidentally in conversations and routines that occur in the classroom.

Key Understanding 1

We can make judgements about order and size without actually measuring. We should think about how confident we can be of our estimate.

Being able to make judgements about order and size without measuring is helpful when actual measurement is difficult or we can tolerate reasonable variations in quantity. We use our perceptual judgement to estimate size, by looking at or feeling things, or experiencing the passage of time. Though a person very familiar with a particular type of material might be able to look at something made from that material and estimate its mass or weight, it would not normally be sufficient simply to look in order to estimate mass; we would need to lift it. A student who tries to estimate the mass of a rock simply by looking at it may well be confusing mass with volume or with some other attribute.

Students should be encouraged to make statements about the confidence they hold in their estimates. A student might estimate a wall to be 7 m wide but claim to be absolutely certain that the wall is between 4 and 10 m wide and pretty sure it is between 6 and 8 m. As students discuss their work, the language of approximation should be clarified (e.g., almost, not quite, a bit less than). They should learn that the suitability of an estimate depends on how confident they would be to use it in particular circumstances. Thus, the suitability or correctness of an estimate depends upon whether it is sensible for the use to which it is to be put and not how close it is to the real measurement.

The focus of this Key Understanding is the development of the following understandings:

- It is possible to estimate a quantity by making a perceptual judgement (that is, by looking or feeling or experiencing).
- We may rely on perceptual judgements of quantity when making a direct physical measurement is difficult or impossible (perhaps we have lost our tape measure or the spot is awkward to get to).
- We may also rely on perceptual judgements of quantity when we are confident that our judgement is good enough for the circumstances.
- To be confident in our judgements, we need to focus on the right attribute and not be distracted by other perceptual features.
- We need to be able to say how confident we are of a particular estimate so we can decide whether it is good enough in the circumstances.

Key Understanding 2 deals with the development of students' skill in making perceptual estimates and Key Understanding 3 with their ability to improve and check estimates by supplementing perceptual judgements with known information.

Links to the Phases

Phase	Students who are through this phase...
Emergent	■ are prepared to make judgements of size in order to deal with familiar everyday matters *For example*: When asked to collect a sheet of paper from the front of the room to cover the top of their desks, students will try to make a reasonable judgement of size.
Matching and Comparing	■ attend to the right attribute to make judgements in familiar situations, distinguishing length from area and mass from volume, although they may not consistently use this language *For example*: Students will pick up the two objects when asked which is heavier or how many of one will balance the other.
Quantifying	■ do not let an overall sense of size (volume) distract them when estimating mass, and neither will they be distracted into thinking that the event that finished last or started first) necessarily took longer ■ understand the use of the language of between to describe estimates, and prompted, will comment informally on their confidence in their estimates
Measuring	■ will say whether they have enough confidence in their estimate to rely on it in particular circumstances, although they may not think to take it into account unless prompted
Relating	■ will say whether they have enough confidence in their estimate to rely on it in particular circumstances, without prompting

Sample Learning Activities

K–Grade 3: ★★ Important Focus

Blanket for a Bear

Have students choose an area large enough for a purpose just by looking. For example, ask students to select a sheet of paper about the right size for a blanket to cover their bear (a wall space to display their work, a pasture for their collection of farm animals). Ask: Can you choose without taking your bear (your work, your collection of farm animals) with you? Draw out the idea that they can often tell by looking.

Sitting Around a Hoop

Ask students to estimate how many of them could sit around a hoop. Invite them to check to see if they are close and then encourage them to modify their estimates. Repeat this with larger circles and other shapes. Focus on how confident they are in their estimates; (*I am certain 10 could fit, and maybe even 15, but 20 would be too many*.) Ask: What did you look at to help you judge how many? Which part of the hoop did you look at? What did you think about when you were looking at the hoop?

Packing Away

While packing away equipment, ask students to estimate volume and capacity. For example, say: Choose a box that all the blocks or balls will fit into. Ask: Are you sure the long blocks will fit in that box? What about the smaller box? Why do you think the long blocks will not fit? Which boxes do you think will definitely not be big enough? What made you decide?

Odd Lids

Have students sort through a box of odd lids to find lids for particular containers. Encourage them to state their choices before testing them. Ask: What sorts of things tell you it should be the right lid? Do you think it will be too big or small? Do we need to try all of the lids on each container? Draw out the idea that looking backwards and forwards from the lid to the container can help us judge if it will be a close fit.

Balancing on the Teeter-Totter

Have students estimate everyday objects by mass. For example, ask: Who could balance you on the teeter-totter? What is the heaviest thing you could carry in that dump truck without it tipping over? What could balance with that apple on the balance scales? Encourage students to explain their decisions. Ask: How could you decide which is heavier? Why is it hard to know which is heavier just by looking?

Being Late

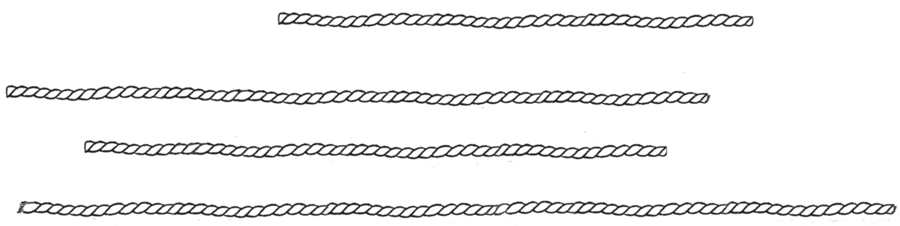
Include students' judgements about time in oral stories. For example, say: Jason told the others, *We will all be late if we go to the other side of the athletic field because the bell will ring before we get back*. Ask: How can you tell when the bell will ring?

Farthest Throw

Have pairs of students estimate how far apart they should stand to play a game involving kicking or throwing a ball to each other. Encourage them to base their estimate on a previous day's experience of throwing or kicking a ball. Ask: How did you decide how far apart to stand ? How can thinking about how far you kicked the ball yesterday help? How could pretending to throw a ball help you to think how far away your partner should be?

Choose a Rope

Have students estimate a length of rope to tie between two posts as a barrier for sports day. Lay out the ropes alongside each other far enough away from the posts to make direct comparison difficult. Say: It is too much effort to try all of these ropes, so pick out two lengths that you think are long enough to tie between the posts. Encourage students to first look at the gap then choose two ropes. After students have checked to see if their ropes fit, ask: What did you look at to help you choose a rope that was long enough? How did you know that rope would be too short?



Sample Learning Activities

Grades 3–5: ★★ Important Focus

How Many People?

Invite students to estimate the room available for classroom activities. For example, ask: How many students will fit in the reading corner (the computer room)? How much room does one person need? How do you know that your estimate is close enough without moving people? Move the furniture to change the size of the space and ask again.

Display Board

Ask students to decide whether they need to measure or estimate in order to work out how many pieces of paper will fit on the display board. Ask: What will you look at to help you imagine how many fit across the board in one row? How many rows do you think will fit down the board? How sure are you that that many pieces of paper will fit? What would you need to do to measure how many do fit? Which would be easier—to measure or to estimate?

Estimate or Measure?

Have students consider different hypothetical situations where a judgement about mass is required and decide whether estimating or measuring would be appropriate. For example, say: It is winter and grapes are very expensive. Would you lift the grapes to estimate how much you wanted to buy, or would you want to check the mass on some scales before you bought them? Why? Why not?

Marking Out Games

Have students estimate to mark out games. For example, say: Let us mark out a hopscotch game. Ask: How will you judge the size for each shape? How long should it be altogether? How wide? Do you need to measure, or can you tell by just looking? Will it make a difference to the game? When setting up for baseball(tee ball), ask: How do you know this ball diamond is a reasonable size? When would we need to measure exactly?

Time Before Recess

Invite students to judge if there is enough time left before recess (lunch) to play a game. Ask: How do you know how long it will take? How do you know how much time we have left? How can you estimate the amount of time left? How can you estimate the amount of time it will take to play the game? (past experience)

Different Amounts of Drink Mix

Ask: How do you decide how much powdered drink mix to use when mixing a drink? Why not measure out the amount? What happens if you use too much (not enough) mix? What makes you feel sure you will have the right taste? Have different students make up some mixtures of drink mix and water and compare the taste. Ask: Why is the taste different in the different glasses? What did you look at when you were pouring in the drink? How did you know when to stop pouring? How could you make sure your glasses all had the same taste?

Covering a Container

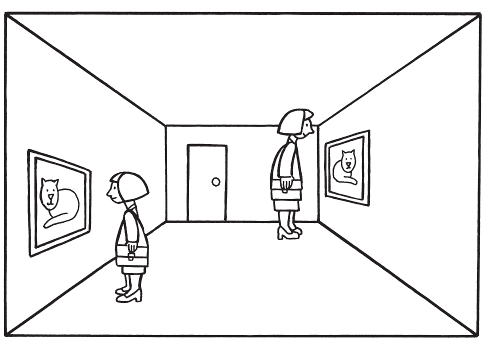
Ask students to estimate and choose the size of paper needed to cover the outside surface of a container to make a pencil holder. Invite them to test their estimate and, if necessary, choose another size. Ask: Why do you think you underestimated (overestimated) the size? Did you use the length (width) of the paper, or the area? Why? If you were using gold leaf paper, would an estimate be good enough? Why? Why not?

Will it Fit?

Have students decide whether they can rely on an estimate to say if things are big enough (about the right size, not too big) for practical purposes (selecting a piece of wrapping paper to cover a present, cutting string to tie around a box, choosing a box to fit all the books in). Ask: Which part of the object are you looking at in order to make your decision?

Distorted Estimates

Discuss factors that distort estimates (time passes slowly when you are waiting for someone, objects look smaller and closer together when they are farther away, a tall narrow glass looks like it would hold more than a short wide glass), including visual illusions. Both line segments below are the same length, but the bottom one appears to be longer because of the direction of the arrows at each end; the person at the back of the room looks taller because the room has been drawn to perspective while the people have not. Both people are actually the same height.



Sample Learning Activities

Grades 5–8: ★★ Important Focus

Track and Field

Have students decide how accurate measurements need to be for different purposes. For example, ask students to think about setting up the school athletic field for the track and field day. They would need to decide:

- what size each waiting area should be for each sports team
- how much tape they will need to mark out the waiting areas and the running lanes
- the position of the markers for team games

Ask: Where might estimation rather than exact measurement be sufficient? What unit of measure would be accurate enough for each job? When do you need exact measures?

Bottles of Drink Mix

Ask students to decide how prepared they would be to rely on their estimates. For example, invite them to estimate how many litres of drink mix are needed for every student to have a glass. Ask: If most estimates were about $1\frac{1}{2}$ mL, how confident would you feel about buying one 2-L bottle? What would you buy if the estimates were between $1\frac{3}{4}$ mL and $2\frac{1}{3}$ mL? Would it be better to over-estimate or under-estimate in this situation? Why? Is an estimate enough to decide or would an accurate measure be better?

Reasonable Estimates

Encourage students to use past experience to judge the reasonableness of each other's estimates. For example, say: Some students said they could walk around the athletic field in one minute. Do you think they could? What would be a more reasonable length of time? How do you know?

Parent Panel

Have students invite parents to form a panel to answer students' questions about when they estimate and measure at work; for example, a builder might describe how and why the amount of mortar needed to lay bricks for a section of wall is estimated and why the placement of the first brick course is carefully measured to millimetre accuracy. Ask: Why is estimating chosen over measuring in some of the situations? In what kinds of situations is measuring important?

Elevator Problem

Have students decide whether we might over-estimate or under-estimate in realistic situations. For example, say: The mass limit given on an elevator is 905 kg. How many trips would it take to carry our class up to the ninth floor? Should we use our closest estimate, an over-estimate, or an under-estimate of students' weight?

Getting to the Bus Stop

Have students decide when an under-estimate or over-estimate of time intervals is needed (getting to the bus stop, playing outside, baking times for cookies, travelling time to school). Ask: If I needed to get to the bus stop to catch the 10 a.m. bus, how much should I underestimate or overestimate the time it would take me to get there? Should I under-estimate or over-estimate the time I spend outside (the time it will take the cookies to bake)? By how much?

Key Understanding 2

We can improve our estimates by getting to know the size of common units and by practising judging the size of objects and events.

This Key Understanding deals with the improvement of students' skill in estimating quantities by making a perceptual judgement. We look at a man and say he is taller than Dad; we look at a jug and say it is big enough to hold two cups of sauce. We lift a rock and say it weighs a bit more than a kilogram, and we experience the passage of time and judge that at least 10 min have passed. Students should understand that even though estimation relies on perception, it is not just guessing. Estimation involves judgement that has improved with the help of experience; that is, with practice.

Practice helps us to become both better at estimating quantities and more confident in our judgement, so that we are prepared to trust it. Helpful practice involves:

- making an estimate
- getting feedback on how close the estimate was (often by measuring immediately)
- consciously using the feedback to improve the next estimate, and repeating the cycle

To estimate several things and then check all of them is less likely to improve our estimation skills. Sometimes, students misunderstand the request to estimate then measure and develop the mistaken view that we would normally do both. They may then think that measuring is better than estimating, and even that an estimate is wrong if it is not the same as the measurement.

Students should be clear that in school the reason we often measure after estimating is to get better at estimating. In real life, they will be expected to do both. We estimate instead of directly measuring. If we have faith that our estimation skills are sufficiently good for the situation, then we will not measure.

Links to the Phases

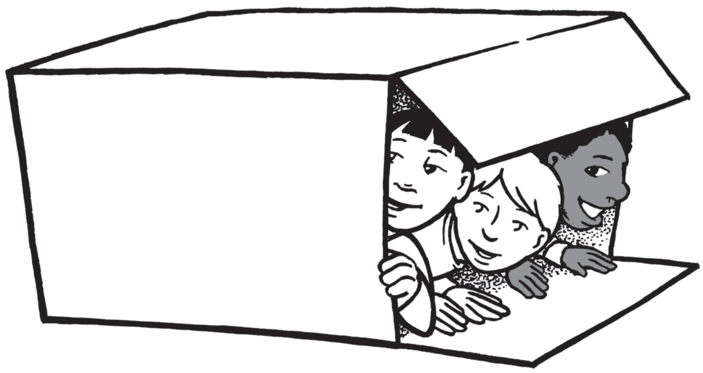
Phase	Students who are through this phase...
Emergent	■ will look to find something clearly longer than an object and lift to find something clearly heavier than an object
Matching and Comparing	■ will generally be able to find lengths, masses, or volumes of objects that are about, less than, and more than one provided ■ will make reasonable length estimates up to about five or six units if they can see and handle a representation of the unit, such as a rod or a walking step
Quantifying	■ are able to find areas and times (to the hour, half hour, and five minutes) that are about, less than, and more than ones provided ■ make reasonable estimates of length, mass, area, volume, or angle up to about six units, if they can see and handle a unit such as metre, litre, and kilogram
Measuring	■ use the known size of common objects (e.g., a litre carton of milk) as benchmarks to assist them with their estimates ■ use the known size of common standard units, such as centimetre, metre, litre, and kilogram, to find objects about that size without the unit actually present
Relating	■ have a well-developed sense of the size of common standard units and can find lengths of about 1 mm, 1 cm, 1 m; volumes of about 1 L, 250 mL (a cup), 25 mL (a tablespoon); masses of about 1 kg, 100 g; and areas of about 1cm^2 and 1 m^2

Sample Learning Activities

K–Grade 3: ★★ Important Focus

Animal Homes

Have students find spaces large enough for one student to use as a home in imaginative play. For example, provide boxes when students are pretending to be animals. Encourage students to try the space. Ask: Is it as big (small) as you thought? Invite some students to pretend to be animal families and look for homes that will fit two, three, or four students. Ask: How did you judge that box would be big enough for all three of you? What did you look at to make your decision? Use the same boxes repeatedly over a few days. Ask: which box fits two (four) people? How do you know?



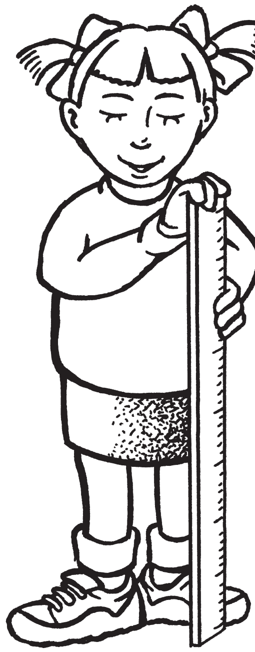
Benchmark

Have students stand in a circle. Ask one student to step forward and ask the others to find a student who is shorter (taller). After several attempts, sit the class down, stand one student up and repeat the activity. Encourage students to visualize the heights of others. Try out each suggestion as it is given and let the student have another attempt to see if they can improve their estimate. Ask: What are you thinking about when you are trying to judge height? What difference does it make when you all sit down? Extend this to objects in the room by asking students to estimate which objects are taller than a selected object.

My Metre

Invite students to find how far up their body a metre is. For example, say: A metre is up to my stomach. Where do you think a metre would come to on you? Ask them to practise imagining a metre high as well as a metre wide.

Ask: Is your bike seat more than 1 m from the ground? Is your desk wider than a metre? Is the width of the doorway more or less than a metre?



Just a Minute

Ask: When someone says, "Just a minute," how long do they mean? What activities do you think would take you about a minute? (putting on socks, getting a drink) Invite students to try out their suggestions as they make them, telling them when to begin, then stopping them after a minute's duration. Encourage the class to use this information to suggest activities that are closer to a minute.

Fingers and Thumbs

Give students pieces of cardboard exactly 1 cm wide and ask them to find parts of their fingers and thumbs that are the same width as 1 cm. Ask them to estimate centimetre-sized lengths in the room and check with their benchmark. Ask: What will happen to the size of your centimetre part when you get older?

Minute Timer

Have students stand or sit with their backs to a one-minute egg timer. Say: I am going to say "Start!" when I start this one-minute timer and I want you to put your hands on your head when you think a minute is up. Set the timer and say: Start! Ring a bell to show when 1 minute is up. Ask: How close were you? Can you get closer? What did you think about to help you decide when the minute was up? Repeat to help students improve their estimates.

K–Grade 3: ★★ Important Focus

A Metre

Have students use a 1-m length of cardboard or string to find different things in the classroom that are close to 1 m (the door knob, the length of the desk). Have each student choose one of these to use as a benchmark for a metre length and estimate whether other lengths around the room are more than, the same as, or less than a metre. Encourage them to check each time so that they can improve their estimates. Ask: What did you think about when you compared the length of your benchmark to the doorway? Why do you think you overestimated (underestimated) the length of a metre on your last try?

Furniture Through the Door

Ask students to decide which furniture will fit through the door so the room can be repainted. Encourage them to categorize the furniture into those pieces that will easily fit through, and those that will be close. Ask: What makes you sure that these will fit through easily? What makes you sure the other ones will not? Which pieces of furniture would you need to measure before you move them? Why?

Vegetables and Fruit

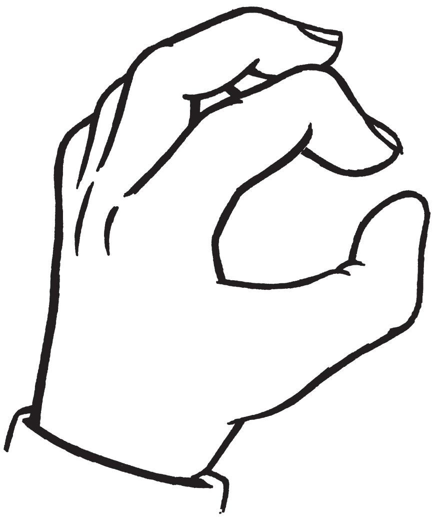
Have students estimate a kilogram of various vegetables and fruits by comparing to a known kilogram benchmark. For example, select a common grocery item that has a mass of a kilogram (a litre of water, a kilogram can of fruit, a kilogram of yogurt) and put it in a bag to use as a kilogram benchmark. Place various quantities of fruit or vegetables in plastic bags and invite students to lift them to find those that match the mass of the benchmark bag. Encourage them to use a balance scale to check each time and thus improve their estimates. Ask: Why do you think different people made different judgements about which bags had a mass of 1 kg? Can you make up a bag yourself that you think contains a kilogram of potatoes? How does having the known kilogram bag help you?

Sample Learning Activities

Grades 3–5: ★★ Important Focus

Thumb and Forefinger

Have students practise holding their thumb and forefinger 1 cm (2 cm, 5 cm) apart and have their partners check with pieces of cardboard cut exactly to each measurement. When they are able to do this with confidence, suggest they challenge family members to match their skill. Encourage students to use this visual memory to estimate the length of small items (an eraser, a pencil sharpener).



How Long Is a Metre?

Have students estimate metre lengths and distances and, with feedback, develop personal benchmarks for the unit. For example, first have students tear off a length of tape that they judge to be 1 m long and then compare the tape to a metre tape measure. Invite students to try again, adjusting their next estimate according to their first result. Encourage them to use their outstretched arms in some way to help judge the metre length. Second, draw two chalk lines exactly 1 m apart and have students step out the distance in various ways. Third, have students use a metre rule to find a part of their body that is one metre above the ground. Ask them to estimate other heights of 1 m and check with their personal benchmark. Ask: How does checking your estimate help you get better at estimating a metre? How did working out a personal benchmark help improve your estimates? Why might you need to measure and check your personal benchmarks in six months or a year? (See Case Study 1, page 105.)

Grades 3–5: ★★ Important Focus

A Litre

Ask students to estimate a litre of water in a bucket, then pour it into a milk carton to check. Encourage them to try again, making adjustments, until the estimate is close to a litre. Then, invite students to estimate a litre of water in a large bowl, testing again with the litre carton. Ask: Why do you think your final bucket estimate was closer than the first bowl estimate? What were you looking at when you first estimated the litre of water in the bucket? How did you improve your estimate the second or third time? What did you think about when you were estimating the litre in the bowl? How could you estimate a litre of water pouring from the tap?

Cans of Food

Have students lift to compare cans of various foods in different sizes with a known kilogram weight or object. Ask them to estimate the number of each type of can that would approximately equal a kilogram. Invite students to use balance scales to check results and try to improve at each attempt. Record and compare successive estimates. Ask: How did lifting the kilogram of jam help you to estimate how many cans of baked beans would weigh 1 kg? What did you think about to improve your estimate after you had checked the balance scales?

Food at Home

Have students find things at home that are packaged in 1-kg amounts (e.g., 1 kg of rice, 1 kg of potatoes, 1 kg of sugar). Ask them to write about the size and feel of each package. Ask: How did the kilogram of potatoes feel different from the kilogram of rice? How would a kilogram of potato chips feel the same or different from a kilogram of sugar? Later, bring in a range of similar packaged things that vary in weight between 250 g and 2 kg and cover the weight information with stickers. Have students examine the packages and then lift them to identify which have a mass of about (less than, more than) a kilogram. Ask: How did looking at the food help you decide which? How did lifting help you choose the packages that were about a kilogram?

A Square Metre

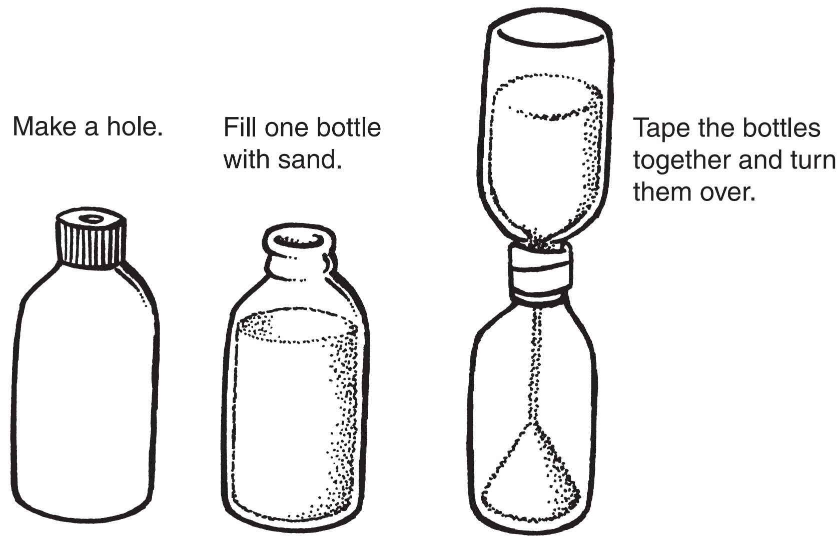
Have students join newspaper sheets to estimate and construct a metre square. Invite them to test their estimate by measuring the length of each side and then adding or removing paper until they have a square with sides of exactly 1 m. Then, ask students to estimate which things in their environment would have an area of about a square metre by visualizing a match with their newspaper square. Ask: How can you judge if its area is a square metre if it is not a square? (imagine cutting up the newspaper square and rearranging it to fit the shape). What are you thinking of (looking at) to say that the long, narrow window is about 1 m^2? Can you imagine a circle that has an area of 1 m^2? Would it have to be wider or narrower than your newspaper square? Why?

A Square Centimetre

Have students visualize how many square-centimetre tiles will cover the floor plan of a room. Encourage them to look at a 1-cm square tile next to the plan and think of ways to judge how many would be needed. Invite them to compare methods, then estimate again using a different floor plan. Ask: Did your strategy change after your first try? How? Were you more confident in your estimate the second time? Why? Why not?

Egg Timer Estimates

Have students make an egg timer with two plastic bottles and help them to adjust it to measure exactly 5 minutes. Organize students into pairs and ask them to take turns estimating 5-minute intervals while their partner checks using the timer. Ask: How close was your estimate? Can you get closer with practice? Does what you are doing during the 5 minutes affect how accurate your estimates are?



Standard Volumes

Give students 1-L drink bottles or milk cartons to use as a benchmark to help them estimate the volume of liquids. Have students pour water into and out of the litre container to help them visualize the amount of liquid in a litre. Invite students to decide whether other amounts of liquid are less than, close to, or more than a litre (a glass of milk, the amount of water needed to water an indoor plant). Ask: What helped you judge that the glass of milk must be less than a litre?

Sample Learning Activities

Grades 5–8: ★★★ Major Focus

Class Charts

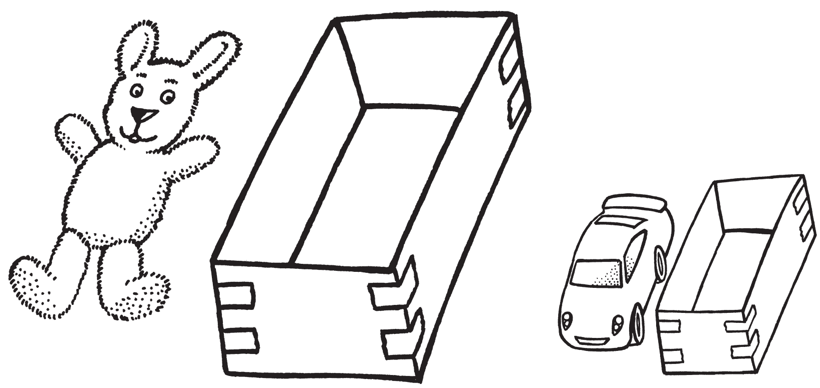
Have students make class charts naming familiar objects that are approximately 1 g, 1 kg, 1 cm, 1 m, 1 mm, 1 L. During classroom activities, encourage students to use the charts to help them estimate other things. For example, say: If you know that bag of rice weighs a kilogram, how heavy do you think that bag of potatoes is?

Length of Pace

Have students develop their own reliable personal benchmarks in order to improve the accuracy of their estimates. For example, ask students to find the average length of their normal walking pace, then the length of different types of paces (striding, jogging, running). Ask: Which types of paces could you use to estimate metres? How reliable would your estimates be using the different paces? Encourage students to choose a pace type to use as their benchmark, and use it to estimate a 10-m distance, a 25-m distance, and a 50-m distance. Have their partner check with a trundle wheel to give feedback on the usefulness of the method.

Box for a Toy

Have students visualize and estimate the size of familiar objects they cannot actually see to measure. For example, say: Think of a small toy or object you have at home and estimate its dimensions to construct a box for it. Suppose all the boxes will be placed in a large container to be sent overseas. Save as much space as you can. When students have made their boxes, ask them to bring in the objects from home and test them in their boxes. Ask: How well did your container match the size of your object? Why do you think you over-estimated (underestimated)? How could you improve your judgement in the future?



Bags of Mass

Have students improve their estimation of mass units. For example, provide plastic bags marked with a range of masses such as 100 g, 250 g, 500 g, 800 g, and 1000 g. Invite students to choose from substances such as rice, beans, sand, or play dough and place what they estimate to be the appropriate mass in a selected bag. Encourage them to check the mass on a kitchen scale and to repeat with the same substance until they get close to the target mass. Invite them to try another mass. Ask: How did you judge 250 g of rice? How was this different from 250 g of play dough? How would you estimate 250 g of rice?

Feedback on Distance

Have students consider the value of different types of feedback for improving estimates. For example, ask students to pace out either 7 m, 9 m, 11 m, or 13 m, then have their partner measure with a trundle wheel and use one of the following four types of feedback:

- right, wrong
- way out, a bit out, close
- much (a little) too long, much (a little) too short
- the actual distance in metres

Have students repeat the estimate, measure, and feedback cycle five times, then choose a different distance and feedback type and repeat the process, with their partners keeping a record of the actual distance paced out each time. Invite students to decide on the type of feedback that best helps improve successive estimates. Ask: Which types of feedback were the most (least) helpful for improving your estimates? Why?

More Feedback

Extend *Feedback on Distance* above to other attributes, such as mass, capacity, time, and angle, to have students find which kind of feedback best helps improve their estimates.

Sorting Shapes and Surfaces

Extend *A Square Metre*, page 100, by having students sort a range of shapes or surfaces according to their estimate of whether they are less than, more than or about equal to one square metre. Include a range of sizes of circles, triangles, rectangles, and other irregular shapes cut out from newsprint or drawn in chalk on the playground. Ask: How did you judge that the triangle was larger than a square metre? How did you imagine cutting and rearranging the narrow rectangle to match a square metre? Is there a different way you can think about visualizing that amount of area?

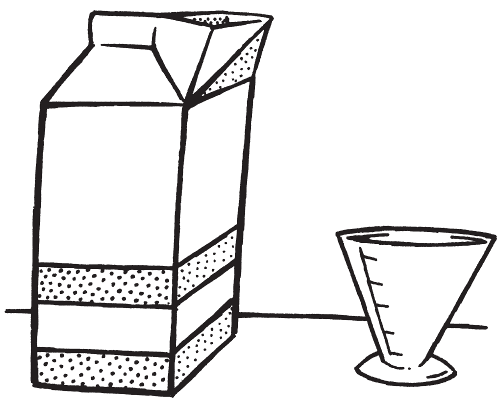
Grades 5–8: ★★★ Major Focus

Length Benchmarks

Have students use a ruler to measure various parts of their hand and forearm to find a reliable personal benchmark for a millimetre, a centimetre, and a decimetre. Invite students to challenge partners to use a straight edge to draw lines with lengths between 1 mm and 30 cm as accurately as possible using only their benchmarks to help them. Ask: Which combination of benchmarks did you use to draw your line 12.5 cm long? How did you judge the half centimetre? Can you find a more reliable benchmark?

Milk Cartons and Mass

Have students fill a milk carton with water and weigh it to determine that a litre of water has a mass of 1 kg. Ask them to use that information and a medicine glass to find amounts of water that have a mass of various small numbers of grams. Then, invite them to pour the amounts of water into small lightweight plastic sandwich bags and lift to gain a sense of the mass of small quantities of water. Encourage them to use lifting to estimate the mass of small objects (pencils, erasers, pencil sharpeners, seeds, small pebbles, table tennis balls). Ask: What is the smallest number of grams that you can reliably estimate? Can you tell the difference between 5 g and 10 g? Why do you think it is so difficult to estimate the mass of very small objects by lifting, compared to estimating mass in kilograms?



Milk Cartons and Volume

Help students establish that 1 L is the same volume as 1000 cm^3 or 1 dm^3. Invite students to estimate the volume of various objects by imagining a milk carton and comparing its dimensions to the size of the object. For example, *I think the volume of the softball would be close to a cubic decimetre, because I can imagine it fitting into the bottom half of a 2-L milk carton*. Ask: How could you estimate smaller volumes by thinking about smaller milk or juice cartons?

CASE STUDY 1

Sample Learning Activity: Grades 3–5—How Long is a Metre?, page 99

Key Understanding 2: We can improve our estimates by getting to know the size of common units and by practising judging the size of objects.

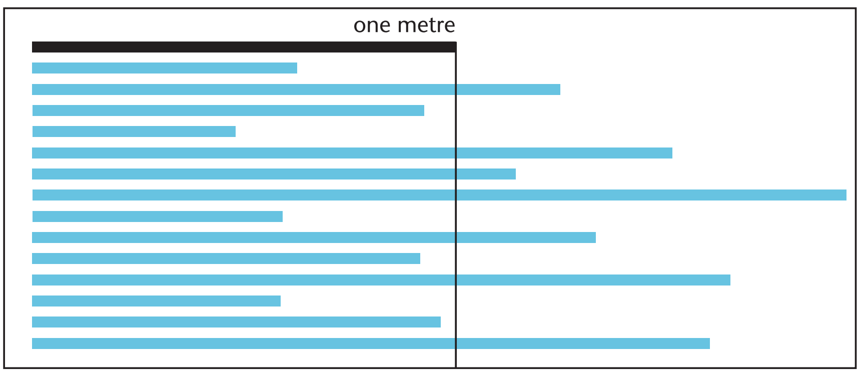
Working Towards: Matching and Comparing Phase and Quantifying Phase

TEACHER'S PURPOSE

My Grade 3 students had often used a trundle wheel to measure outdoor distances in metres and a metre rule to measure the width of things in the classroom. When I asked my students to estimate the distance to the water fountain in metres, however, they all simply counted the steps they took. So I decided to set up some activities to help them develop personal benchmarks for a metre.

ACTION AND REFLECTION

I gave the students rolls of paper tape and asked them to tear off a strip that they estimated to be a metre long, and write their name clearly on it. The students pinned the strips to a display board, lining up one end. I asked the students to write down whose tape they thought would be closest to a metre and then pinned a metre tape measure above the strips and drew a line down from the metre point across the tapes.



The students looked at and commented on their estimates.

"Oh, mine was much too little."

"Mine is about 2 m long."

"Look, mine is just about the same as Lian's and only a little bit less than a metre."

OPPORTUNITY TO LEARN

I then asked the students to have another try at tearing off a metre of tape. Each new strip was pinned on top of the old strip and the class compared the new estimates to the metre. Most students had overcompensated in their second attempt—if their first estimate was shorter than a metre, their second was much longer than a metre, and vice versa.

Several students wanted to try a third time and I asked what they thought they might do to make it a better estimate.

"The first one I just kept unwinding until I though it must be a metre, but it turned out really, really long. The next one I looked at the tape and thought if it looked like the one on the wall, but it ended up too short. This time I am just going to try and look at it and think it is a bit longer than the wall one," said Choon.

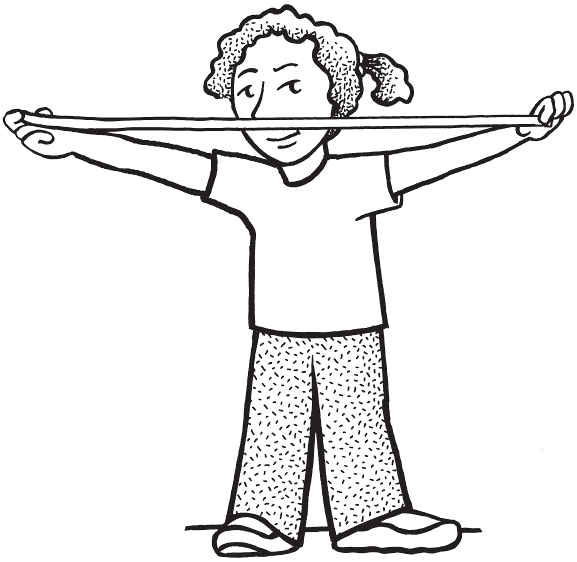
Maria said, "I just guessed the first ones, but this time I am going to put out my arms and think about it."

CONNECTION AND CHALLENGE

After Maria's comment, I told the students how I estimate a metre of fabric or ribbon. I hold one end with the fingertips of one hand and, with my other hand, stretch the ribbon to the tip of my nose with my head turned away.

The students were very interested to watch me tear off a strip of tape using my method, and they were most impressed to see the tape was so close to a metre when tested.

The students were then anxious to try again for themselves, and this time I was pleased to see that most made some sort of physical estimate, and I was able to encourage them to keep trying until they found a way to match a metre.



When students became confident that they had a personal benchmark for estimating a metre, I asked them to go outside and each draw two chalk marks on the playground that they thought were 1 m apart.

I was surprised to find that in this different context most students were unable to use the method they had previously developed and simply took a step, or reverted to guessing. When they paired up to test each other's estimate with a trundle wheel, they were perplexed to see how different their metres were. Several of the students who had taken one step and measured that, were surprised that a metre was so much longer. Sam even accused his partner of turning the wheel twice—he had difficulty believing that the circumference was so much longer than the diameter of the trundle wheel.

I realized that estimating distance was more complicated because they needed to think about some imaginary straight line running between the two chalk marks, then mentally measure this.

I then asked students to find one of the metre paper tapes they had previously estimated with some accuracy and compare this to the distance they had marked out, and to the trundle wheel.

DRAWING OUT THE MATHEMATICAL IDEA

This provided an opportunity to talk about the different things the students had to think about when estimating a metre in situations where there was no line or edge to guide them.

To help them develop a way to pace out a metre length, I drew two lines across the hall exactly 1 m apart, close to our door. Students enjoyed using this guide to find their own way to reliably judge a distance of 1 m.

After several days, I again asked students to estimate a metre distance in the playground and found most had developed some way to approximate this very well.

Key Understanding 3

We can use information we know to make and improve estimates. This also helps us to judge whether measurements and results are reasonable.

This Key Understanding deals with the development of students' ability to supplement their perceptual judgements with known information in order to make or improve estimates and judge the reasonableness of estimates and measurements made by others.

There are a variety of ways in which known information might be built into an estimate.

- Students could make a direct perceptual comparison with something they know the size of; for example, they could estimate the mass of an object by holding it in one hand and holding one or more 50-g chocolates in the other.
- Students could use something they know the size of as a measuring instrument by marking off; for example, they could mark off hand spans across a table, and use their knowledge of the length of the hand span and a computation to estimate the table width in centimetres.
- Students could use ratios or fractions to estimate the size of small things; for example, they could weigh a ream of paper and use it to estimate the mass of one sheet of paper.
- Students could pool a combination of known information and good guesses to estimate quantities without collecting actual data; for example, they could work collaboratively to estimate the quantity of water used in their school each day or the number of kilometres they walk each week.
- Students could average a number of estimates to get an improved estimate; for example, they could average class members' estimates of the same span of time.
- Students could use common events to estimate amount of time and time of day; for example, they could use how busy the parking lot is to estimate how long it is to the end of the school day.

Students should also learn to call upon sizes they already know or to reason on the basis of familiar or known quantities to judge the reasonableness of a result; for example, could the bread really weigh 3.4 kg, or the average height of women in Canada really be 217 cm?

Links to the Phases

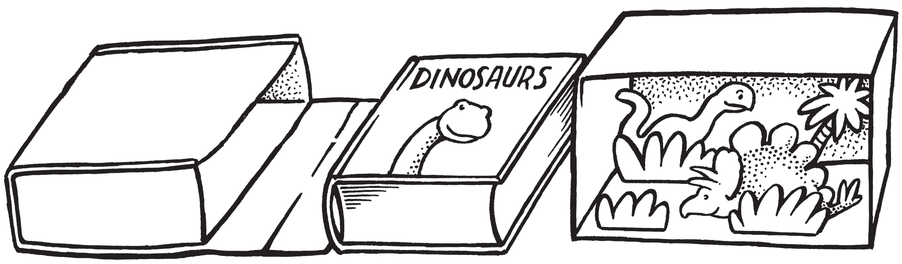
Phase	Students who are through this phase...
Quantifying	■ can identify body parts of about 1 cm, 10 cm, and 1 m, and use these directly to make estimates of length ■ build given information into their judgements *For example*: When told that the door is 2 m high, students will say that the ceiling is about half as much again, so it is about 2 m high. Students will say that since the lunch break is 3 min and the concert fit well within the lunch break, the concert cannot have lasted more than 30 min.
Measuring	■ recall the size of some body parts and movements (hand span, finger width, arm length, walking step) and use these and simple computations to make estimates (*My step is about 90 cm and the garden is 24 steps long so…*) ■ collaborate with others to develop strategies for making sensible estimates of quantities, such as how much water is lost from dripping taps each week at school
Relating	■ have a repertoire of reference points, such as the size of a sheet of paper, and, unprompted, build these reference points into their estimates and their judgements about the reasonableness of measurements

Sample Learning Activities

K–Grade 3: ★ Introduction, Consolidation, or Extension

Using Known Measurements

Encourage students to use known lengths (volumes, masses) of objects to estimate other lengths (volumes, masses). For example, invite them to use something that just fits in their bag as a guide to decide if something else will fit without having to get the bag. Ask: You know that big thick book will only just fit into your bag, so do you think your diorama will fit without squashing it? How did you decide? Or, ask them to establish how many craft sticks fit along their desk, then look at the length of the bookshelf and decide if it is more or fewer craft sticks wide. Ask: How many craft sticks long do you think the bookcase might be? How did you decide?



My Metre

Extend *My Metre*, page 96, by having students use their knowledge of a metre in relation to their body length to judge heights. For example, ask them to use their knowledge of where a metre comes to on their body to decide if heights in the playground are more than a metre or less than a metre. Ask: How can you tell that the monkey bars are more than a metre from the ground? Do you think that the coat hook is higher than a metre from the floor? How can you tell?

Sample Learning Activities

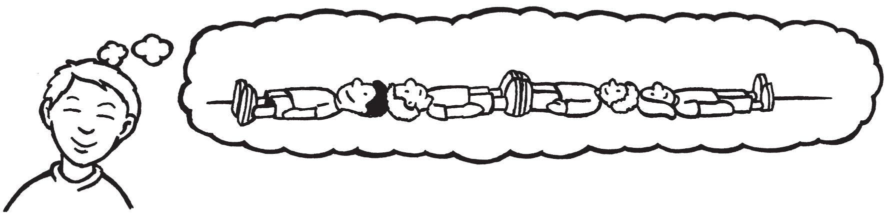
Grades 3–5: ★★ Important Focus

Wanted

After measurement activities that establish the heights of students in the class, present a "wanted" poster with a description that includes the height of a wanted person. Have students identify possible wanted persons from other classes by estimating their heights based on their own height and the known heights of others in the class. For example, *The wanted person is 20 cm taller than me. I know that Katie is 10 cm taller than me and Ian from next door looks about 10 cm taller than Katie, so Ian could be the one*.

Estimating with Heights

Have students use what they know about their height to estimate other lengths (e.g., the height of the door, the height of the window, the length of the room). Ask: How did you decide that the door is twice your height? Can you visualize how many people of your height would fit lying head-to-toe along the wall?



Smallest to Largest Capacity

Have students compare a variety of cups with a standard measuring cup (250 mL), estimating to line them up from the smallest to largest capacity. Ask them to measure to find how many millilitres each one holds and order them again. Invite students to look at other liquid containers and say whether they hold more or less than 250 mL or about how many millilitres they hold. Discuss strategies for making the comparison. Ask: How does the shape of the container affect your estimate? How can visualizing changes in shape help you to compare the container to the measuring cup?

Orange Juice for Lunch

Extend *Smallest to Largest Capacity* above to estimating quantities of liquid for real purposes. For example, ask: How much orange juice would we need for a class lunch? Would 10 L be a sensible estimate? How could you decide from what you know about the capacity of a cup and 1 L?

Grades 3–5: ★★ Important Focus

Fish Sizes

Have students use parts of their body as benchmarks for practical estimation. For example, display a local fisheries chart showing minimum sizes of fish that you are allowed to keep. Ask students to work out personal benchmarks for the minimum sizes of a range of local fish (trout, perch, bass, salmon). Give groups of students different-sized cardboard cut outs of the fish so that they can use their benchmarks to judge which they can keep and which must be thrown back. Appoint one member of each group to be a wildlife inspector and check with a ruler if any under-size fish have been kept. Have students refine their benchmarks and try again to improve their estimates.

The Length of the Track

Have students use measurements they know to judge the reasonableness of estimates. For example, say: Someone told me the length of the track was 200 m, but I am not sure. Ask: How close do you think this estimate might be? How do you know? Encourage students to identify a known 10-m length from which to judge the claim. For example, *I know I can throw the beanbag about 10 m, and it takes about five throws to cross the athletic field. Or, I can see that the track is not much more than twice its width, so 200 m is too much. I think it is only about 100 m.*

Capacities

Have students make use of capacity measures of smaller quantities to estimate large amounts. For example, ask: How much water do you think we will need to refill the fish tank? Think about the capacity of other containers that you already know about, like 2-L ice cream containers or 2-L orange juice containers. How could that help you estimate? (See Case Study 2, page 116.)

Time Schedule

Have students use estimation to work out a time schedule for an excursion. By comparison to known or measured time intervals (how long it takes to eat lunch, get a drink of water, get to swimming lessons by bus), determine how long is needed for the bus trip, how long for lunch, how long to see the display (do the activity), and so on. For example, ask: If it takes about 15 minutes in the bus to get to the pool for swimming lessons and the museum is twice as far away, how long should we allow for the bus trip? Say: It takes about 15 minutes to eat lunch at school and about 15 minutes to get a drink. Ask: How long do we need to allow to get out of the bus and find a place to sit? So, how long do we need for lunch altogether?

Sample Learning Activities

Grades 5–8: ★★★ Major Focus

Marking Off Lengths

Have students use the idea of marking off to estimate a short length from a known long length. For example, say: This bolt is 20 cm long. Estimate the length of this shorter bolt. How does knowing that the short bolt fits along the long bolt about four times help you estimate the length of the short bolt?

Water Used in the School

Have students combine measuring and estimation to solve complex problems. For example, ask them to estimate how much water the school uses in a day. Encourage students to think about what to measure and what to estimate. Stimulate thinking with questions like: How many students are there in the school? How often do they have a drink of water? How can we decide how much one student drinks? How much is it likely to vary? What other ways is water used in the school? Encourage students to consider other sources of information that could help them (e.g., the water company, the cleaners, the gardener, the canteen manager).

A Kilometre

Have students develop personal benchmarks for a kilometre by using a local map to identify a familiar landmark that is a kilometre away from their homes. Ask them to estimate the distance from the school to various destinations based on what they know is a 1-km distance from their home. Then ask them to plan a 5-km jogging circuit around the neighbourhood, using estimation to judge the appropriate distance.

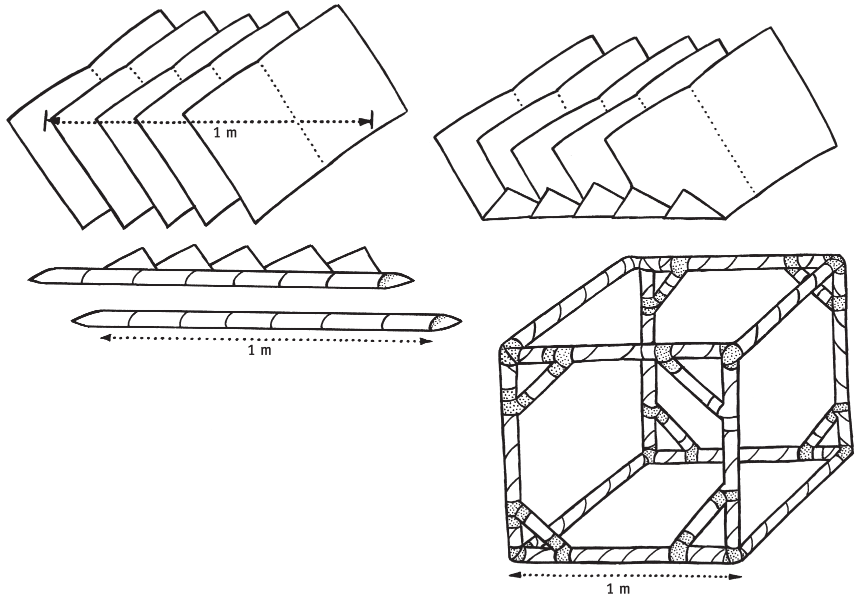
Carpeting the Classroom

Have students use their known height to estimate area. For example, say: The carpet shop has a special on 12 m^2 of carpet. Is this enough for our room? Invite students to say how long and wide the room is using body lengths and use what they know about their height to say if this is enough carpet. Ask: Whose body length could be used as a measure? Did visualizing how many times your body length fits the length and width of the room help you estimate how many square metres the room is? How? Can you draw a diagram to show how it helped?

Grades 5–8: ★★★ Major Focus

Volume of the Classroom

Ask students to estimate in a variety of situations where they do not have access to the measuring unit. For example, invite them to use the image of a metre cube to estimate the volume of the classroom. Ask: Can you use your hands to help you visualize the width and height of a metre cube? Can you imagine how many you would stack to reach the ceiling and how many would fit side-by-side along the wall? How would that help us estimate how many would fill the room? Construct the skeleton of a 1-m cube using rolled newspaper. Ask students to imagine how many cubes would cover the floor and how many would stack to reach the ceiling. Compare this result to their initial estimations. Ask: What have you learned that you can use when you estimate the volume of the storeroom?



Cubes and Litres

Extend *Volume of the Classroom* above by having students estimate the amount of water in a swimming pool. Help students use the connection between litres and cubic decimetres to establish that 1000 L is equal in volume to a cubic metre. Encourage students to look at their 1-m cube and use the kind of thinking that helped them estimate the volume of the room to estimate the number of litres of water in the pool. Ask: How would imagining how many cubes would stack together to fill the pool help you estimate the volume of water in the pool?

Eye Level

Encourage students to use known lengths of body parts to help estimate lengths. For example, invite them to measure their eye level from the floor and use this information together with the image of their ruler to estimate the heights of other people and things. (*My eye level is 126 cm and when I am looking straight ahead at you I see the tip of your nose. I think from there to the top of your head is about two thirds of my ruler; that is 20 cm, so I think you would be close to 146 cm tall*.) Invite students to test their estimates against actual measurements.

Marking Off Cupfuls

Invite students to mark off cupfuls of liquid in order to select the container that will hold the most from several different-shaped containers. Encourage them to put one cup of water in each container and place a mark at that level. Then, have them use the one-cup level to estimate how many more cups would fit in each container by visualizing each extra cup level. Use this information to express the capacity of each container as a range (e.g., between six and seven cups). Ask: How does the shape of the container affect your estimates? What happens to the height of each level when the container gets wider?

Capacities

Extend *Marking Off Cupfuls* above to estimating the capacities of larger containers (e.g., fish tanks, water cooler bottles). For example, pour a 10-L bucketful of water into a fish tank and ask students to visualize the levels of water that additional bucketfuls would reach. Ask: How can we use the level of the first bucketful to estimate the total? Why not just put in 1 L of water and judge from that? (See Case Study 2, page 116.)

Inaccessible Lengths

Extend the marking off idea to estimate a long inaccessible length from a short length (height of trees, height of tall buildings, length of a ship). For example, give students a photograph of a person standing next to a building and invite them to visualize and mark off the “person heights” on the side of the building. Use the known height of the person or the approximate height of people to estimate the height of the building.

CASE STUDY 2

Sample Learning Activity: Grades 5–8—Capacities, pages 112 and 115

Key Understanding 3: We can use information we know to make and improve estimates. This helps us to judge whether measurements and results are reasonable.

Working Towards: Quantiying Phase and Measuring Phase

TEACHER'S PURPOSE

The fish tank in my Grade 6 class needed to be cleaned. To decide how much conditioner to add to the water, the class needed to know the approximate capacity. I decided to use this to help students think about how they could use the known capacity of other containers to estimate unknown quantities.

MOTIVATION AND PURPOSE

I asked everyone to try estimating the quantity of water needed and record this in their math journal. Their estimates varied from 50 to 200 mL and from 1 to 5 L, which suggested that most students had little sense of the size of the units and were simply guessing.

Though students can become very good at recognizing particular sized millilitre and litre containers in common use, they do not consciously make comparisons between them, nor do they often have strategies for using known volumes to judge the capacity of less familiar containers. They need help to make such connections.

OPPORTUNITY TO LEARN

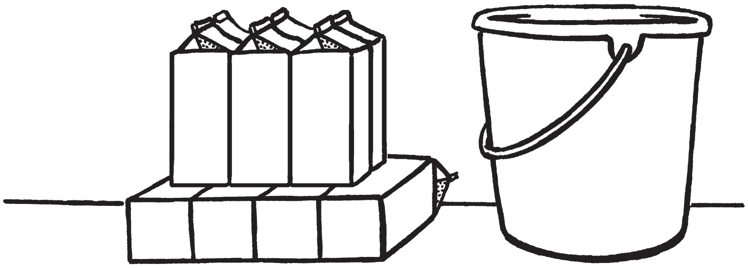
I was certain all students had experience of 1-L carton of milk, a 2-L bottle of pop, and smaller fruit juice cartons measured in millilitres, and I knew some students could quite accurately name the capacity of many of these common containers. But they seemed not to have considered comparing the size of these known containers to the fish tank. For example, Katie said, "It would hold a lot. I bet a hundred mils [sic]."

I asked them to bring a range of liquid containers from home and the class sorted them from the smallest to the largest capacity.

ACTION AND REFLECTION

The students noticed that some different-shaped containers held the same amounts. "The small ice-cream container and the milk carton are both one litre. I thought that the milk carton was bigger, but they are the same," remarked Dion.

The students then used the litre containers to find the capacity of other containers. During these activities, they discovered that the plastic buckets held 10 L. They also stacked 10 milk cartons next to a bucket to see that it did look to be about the same amount, even though it was a different shape.



I asked the students to go back to their estimates of the fish tank to see if they still thought their estimate was reasonable. All rejected their first estimate and were very keen to make another, more informed one.

With very little prompting, most students used what they had learned about the capacity of a bucket to make more sensible estimates.

This is an example from Katie's math journal:

> *I first of all said that the fish tank would hold 100 mL because I thought a hundred was a lot. But 100 mL is not even a little chocolate milk carton. I think it would hold about 20 L because you can see it is bigger than a bucket and a bucket is 10 L.*

Aaron wrote:

> *It had to be more than 5 L because that is just half a bucket and the fish tank is much bigger than that. I think you would have to tip in about 5 buckets of water to fill it so I think it would hold about 50 L of water.*

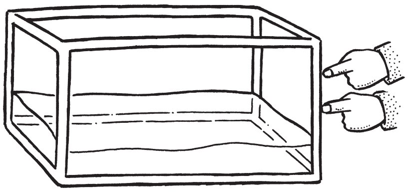
CONNECTION AND CHALLENGE

I then asked the students what they could do to be more confident about their estimates without filling the fish tank and measuring exactly how much water they used.

Biko suggested: "Why do you not just pour one bucket in? We could see how high that goes up and look and think how many more buckets would go in."

Rebecca remembered the pile of milk cartons they compared with the bucket and said: "We could get some empty milk cartons and stack them to work out how many would be about the same size as the fish tank."

Estimate KU 3



The class worked on Biko's suggestion and, after the first bucketful went in, Biko used his fingers to mark the levels he visualized for each bucketful.

I asked if anyone thought they could make a closer estimate than between 30 and 40 L. Jeremy suggested that it was between 30 and 35 L because, "It does not look like any more than half a bucket more would go in."

Kira added, "But it does not look like it could be much less than half a bucket either, so it is more likely 35 than 30 L."

The class then moved on to Rebecca's suggestion. Although she had originally intended that they actually build a pile of milk cartons in the shape of a fish tank, the class now realized they would not have enough cartons in the classroom for this, and they did not really need to do all that work. Instead, I helped students visualize that eight cartons on their side would cover the base of the tank and then four layers would bring this nearly to the height of the tank, arriving at an estimate of 32 L. Students were delighted that the two estimates were so similar, and could see that the difference could be accounted for by the fact that four layers of milk cartons would be a little short of the full depth. Because we would not be filling the fish tank right to the top, the class decided to use the lower estimate to calculate the quantity of conditioner needed.

DRAWING OUT THE MATHEMATICAL IDEA

I thought that students were now beginning to understand that making reasonable estimates was not about making lucky guesses, but often requires them to use some careful reflection and a procedure to compare the quantity to be estimated to an appropriate benchmark. I asked them to take a few minutes to write in their journals what they need to know and do to make good estimates. Their responses suggested they had grasped the main idea, for example:

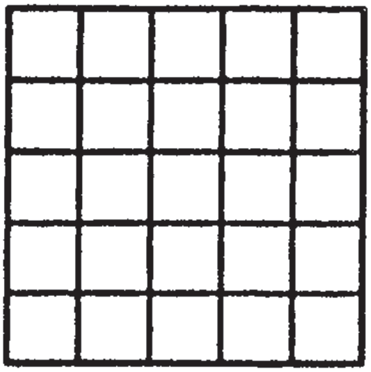
> *You cannot just guess, you have to have something that you know about in your brain and think about how that fit in with what you want to estimate. You think out how many fit in and then you multiply it.*

> *You might want to think how much water and in your mind you look at how many milk cartons would fit in, or how many buckets because you know how much is a milk carton or a bucket, and then you can figure it out.*

Appendix

Name: ______________________________ Date: ______________________

Line Master 1 **Rectangular Arrays**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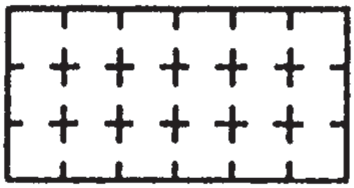
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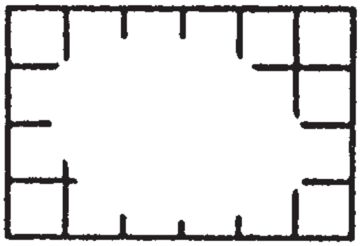
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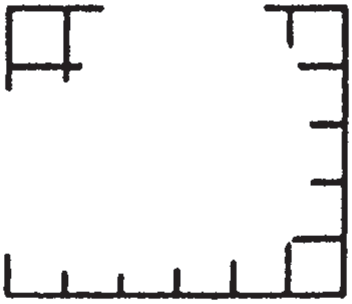
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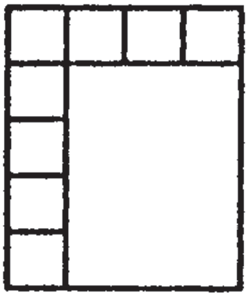
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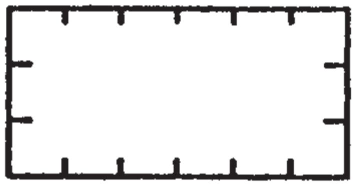
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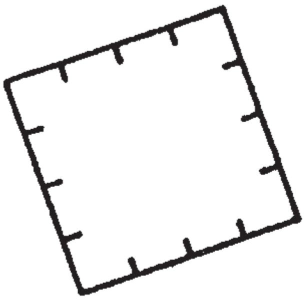
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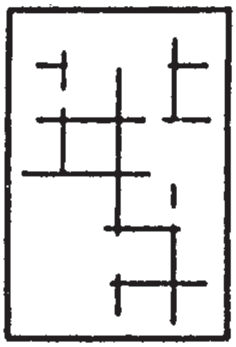
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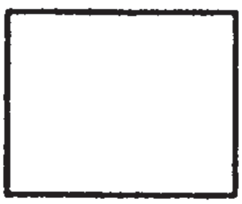
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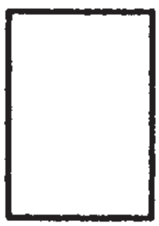
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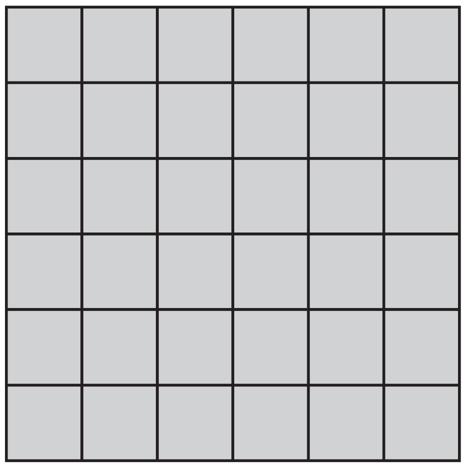
Line Master 2 **1-cm Tiles**



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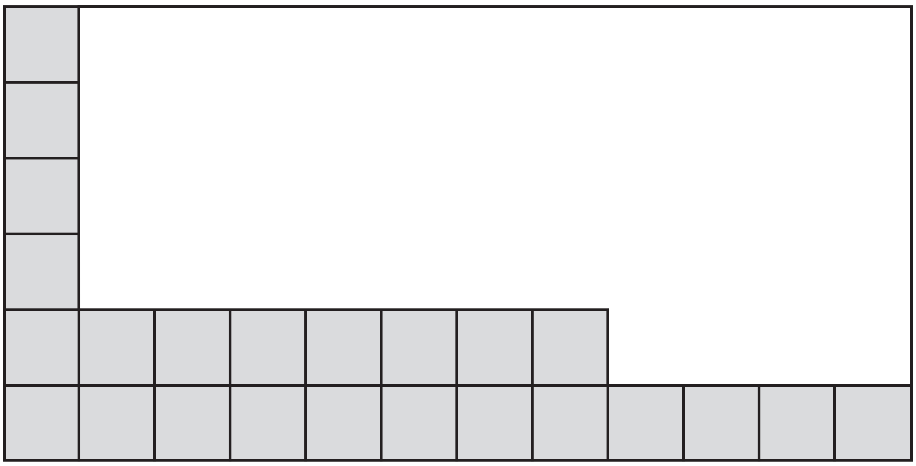
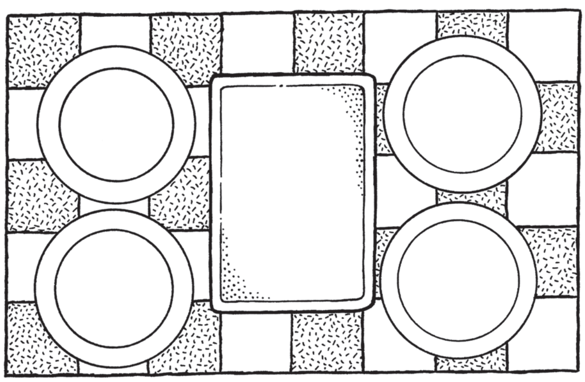
Line Master 3 **Incomplete Grids**





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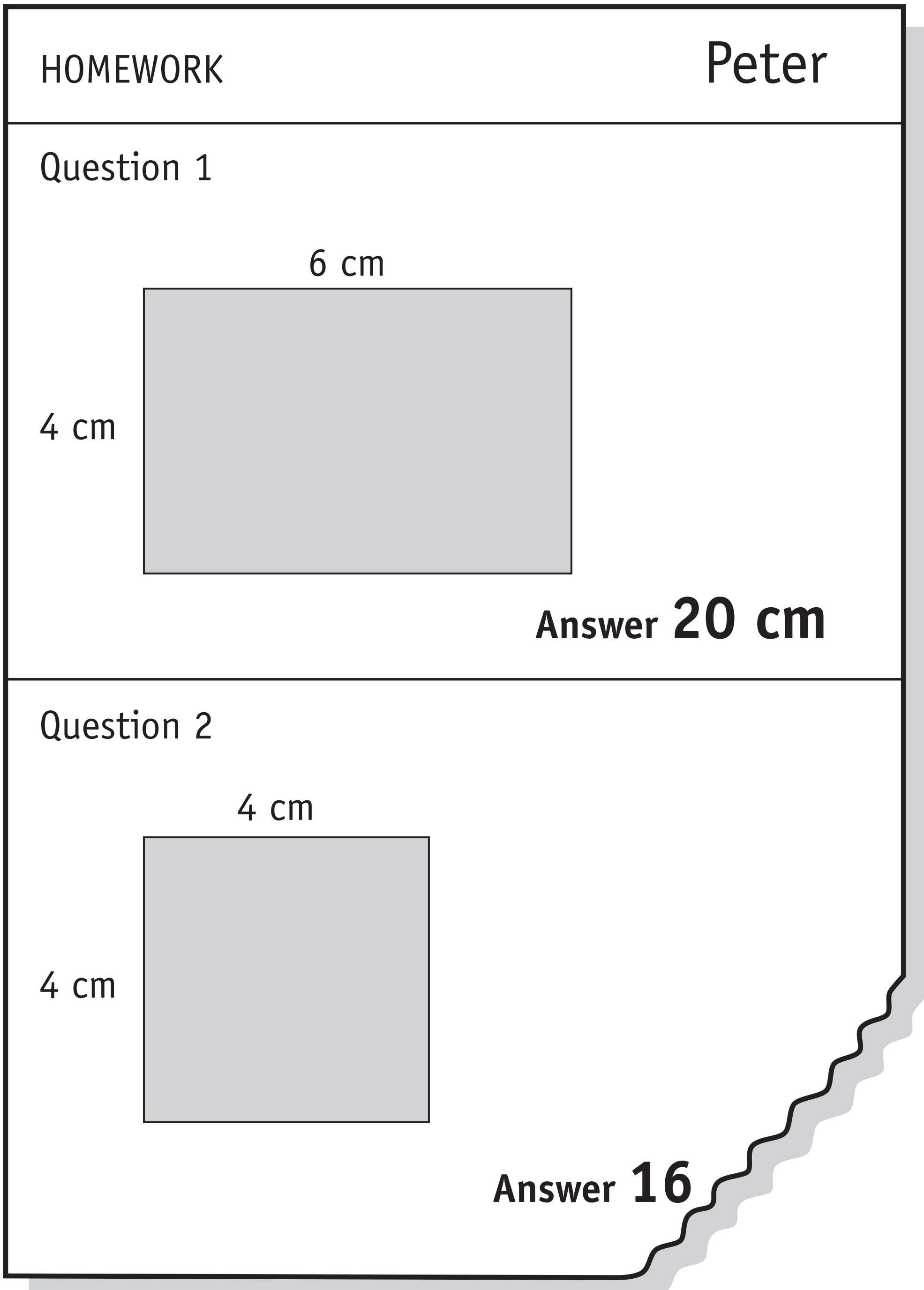
Line Master 4 **Perimeter or Area?**



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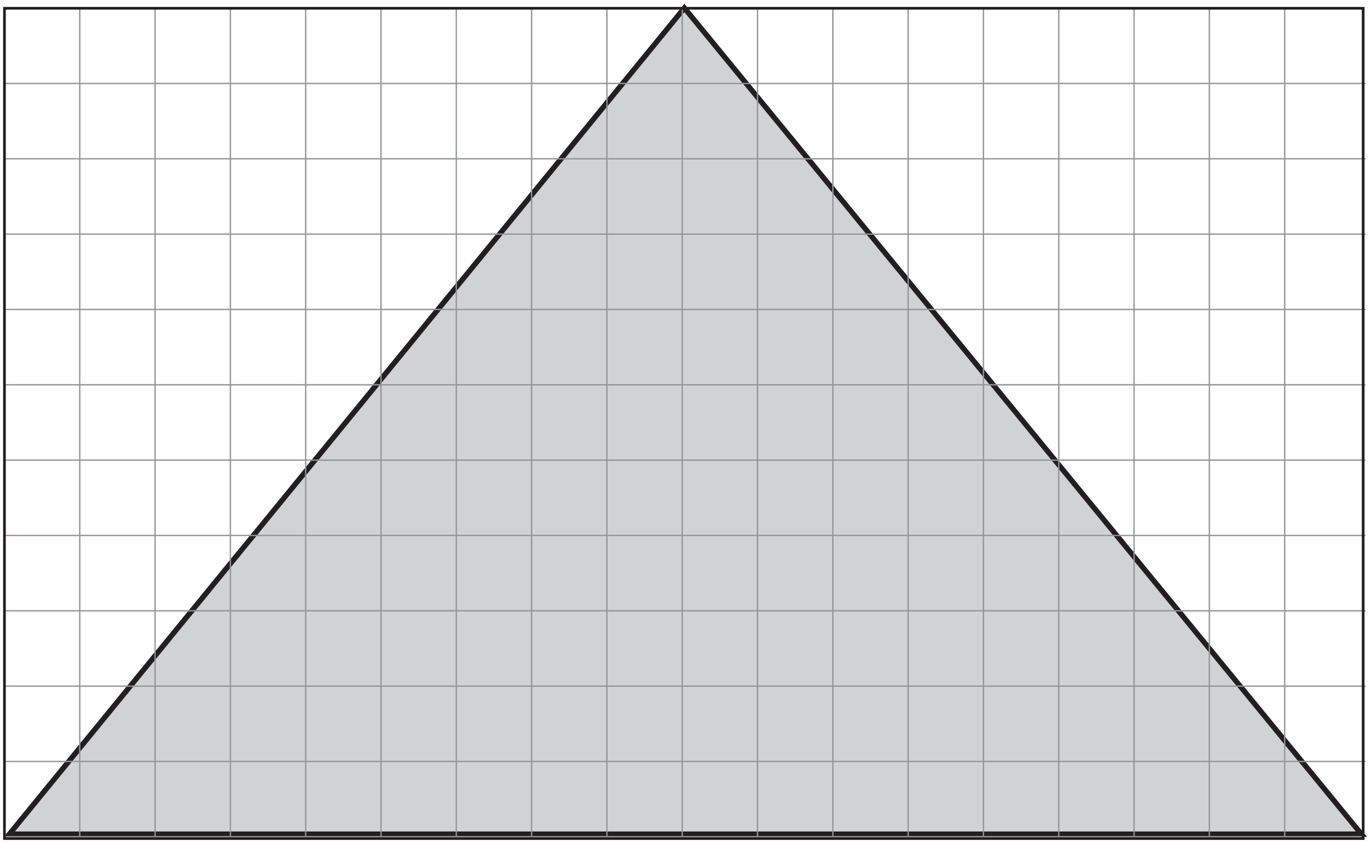
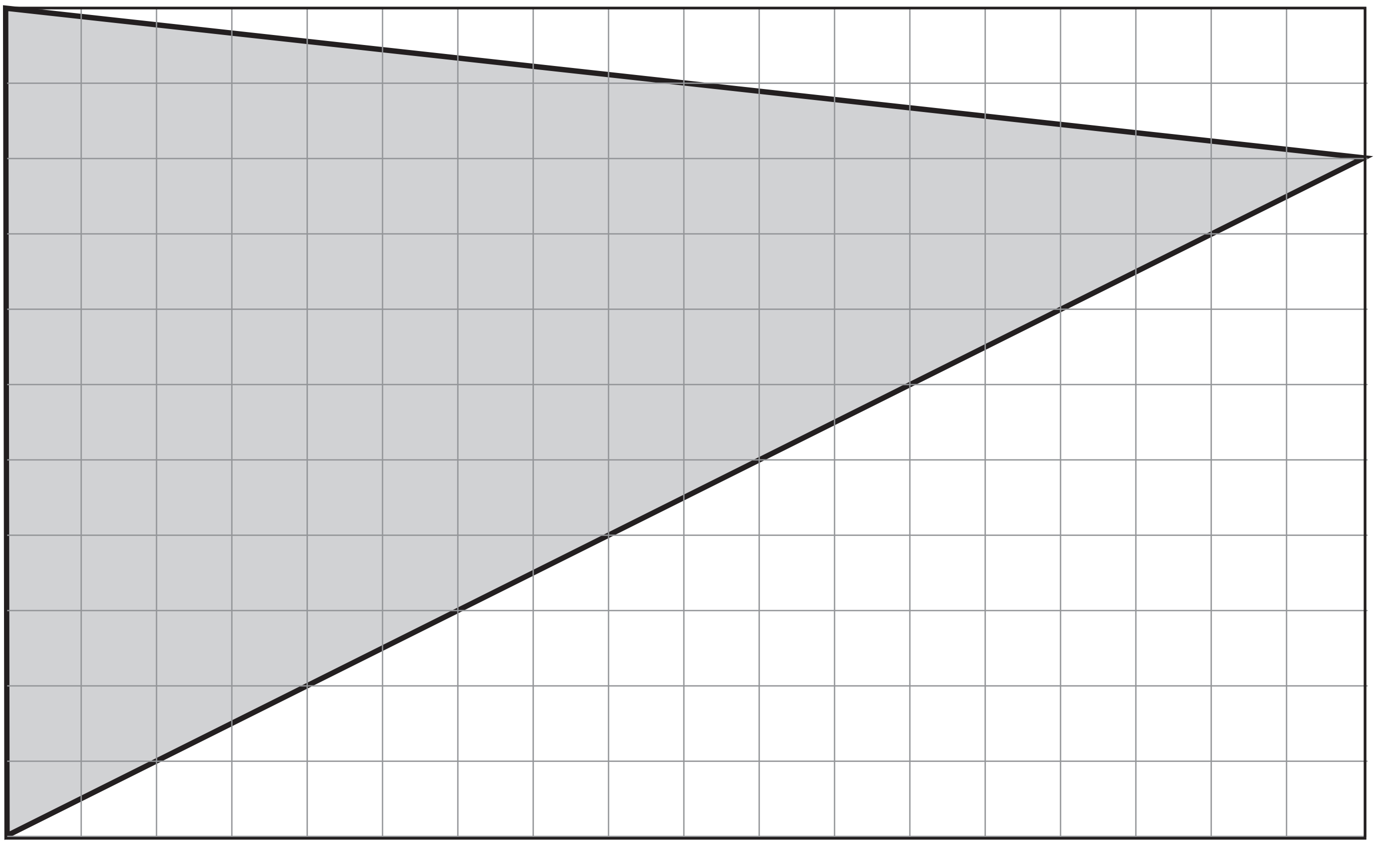
Line Master 5 **Triangle in a Rectangle**





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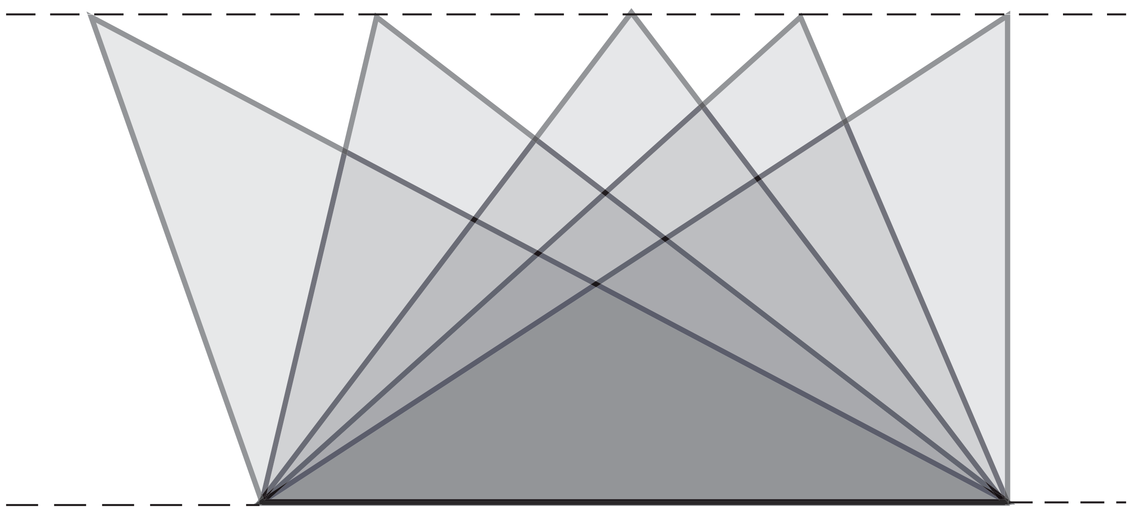
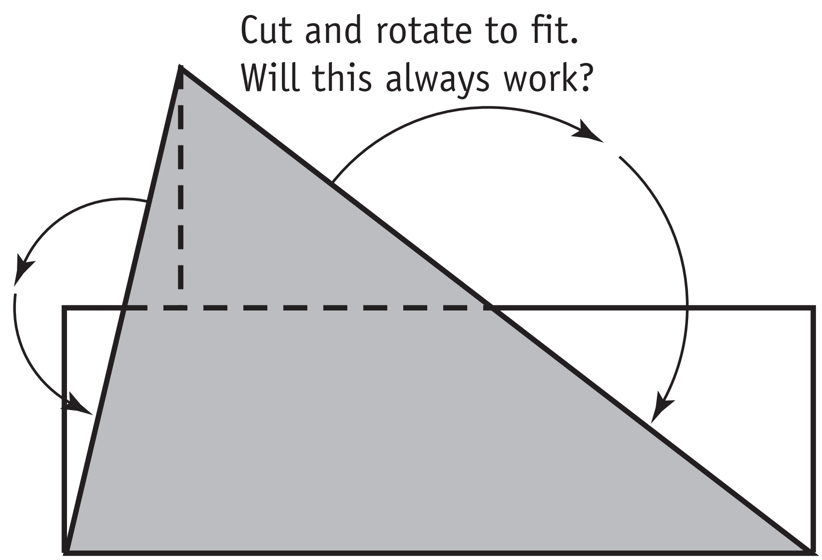
Line Master 6 **Five Triangles**





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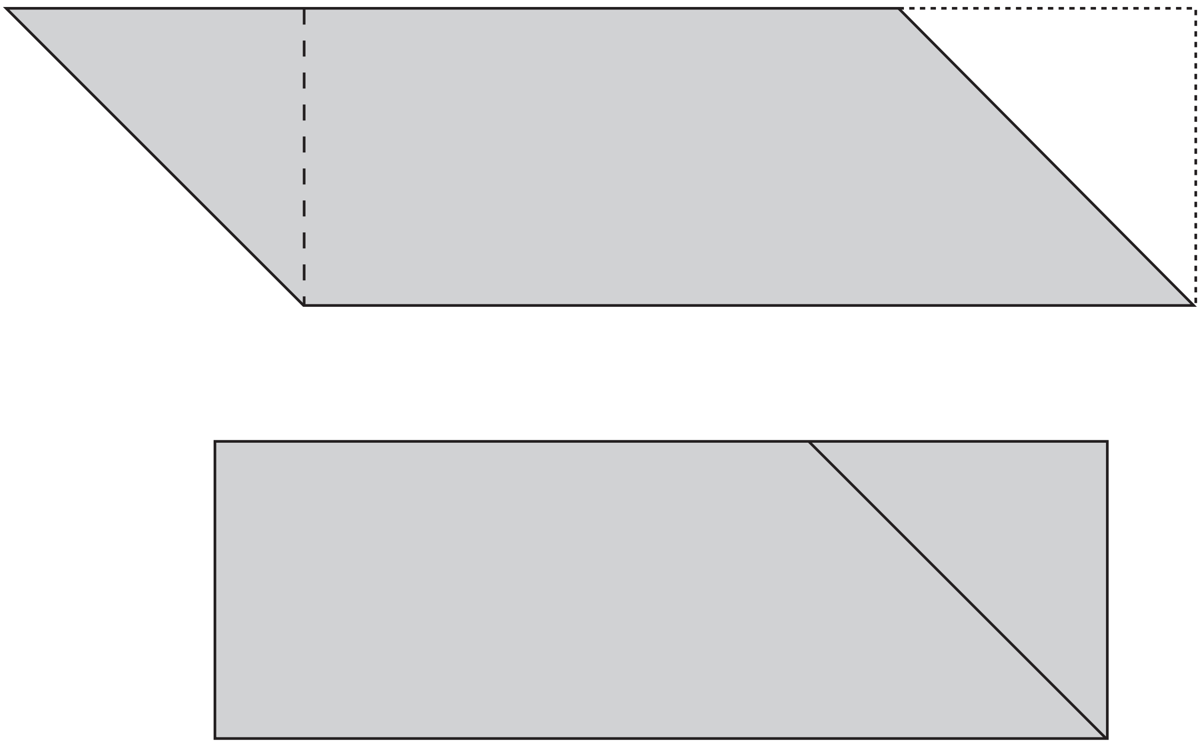
Line Master 7 **Rearranging Parallograms**



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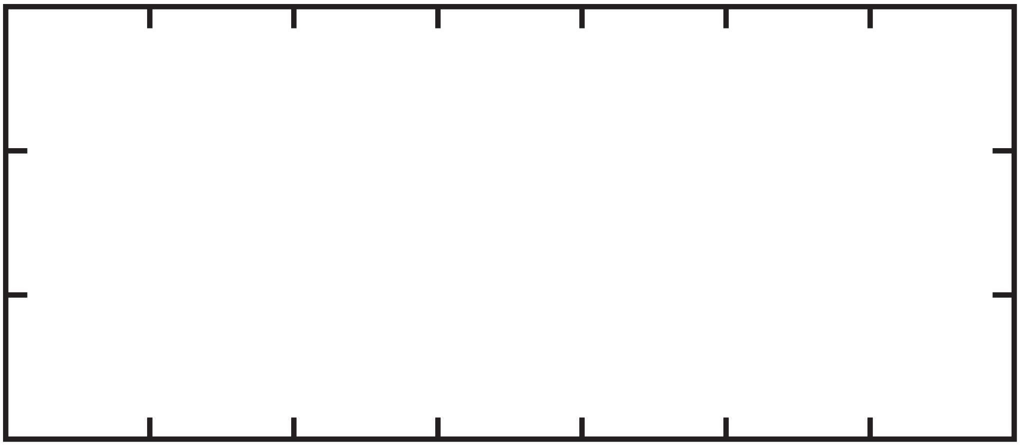
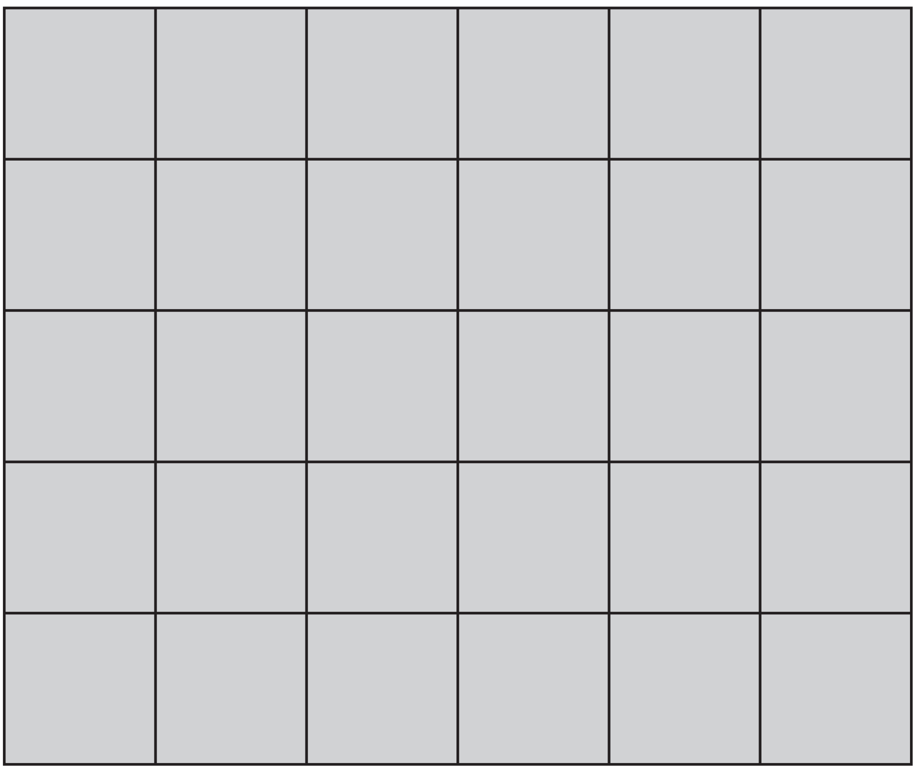
Line Master 8 **Visualizing Arrays**





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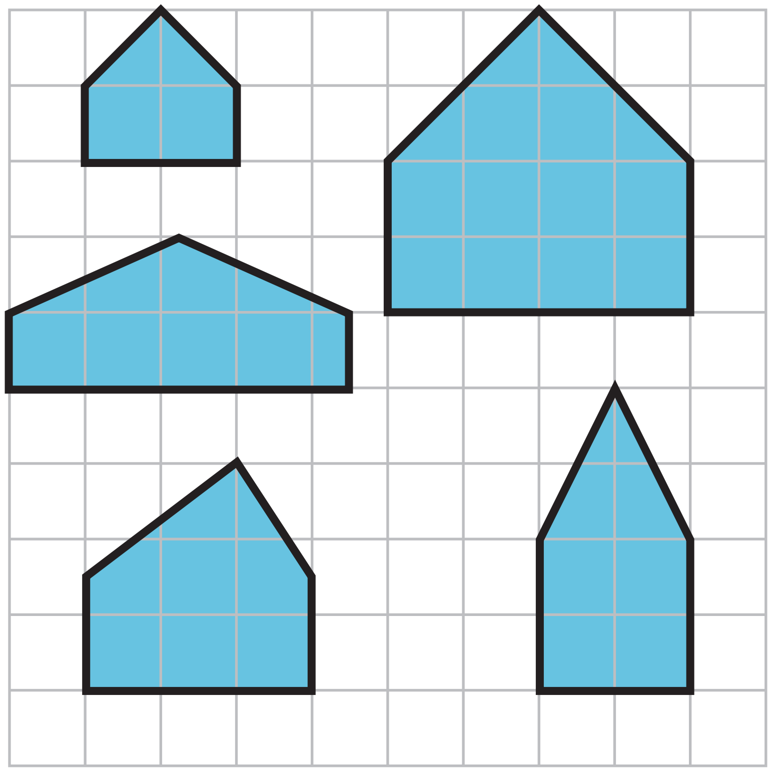
Line Master 9 **Figures on a Grid**



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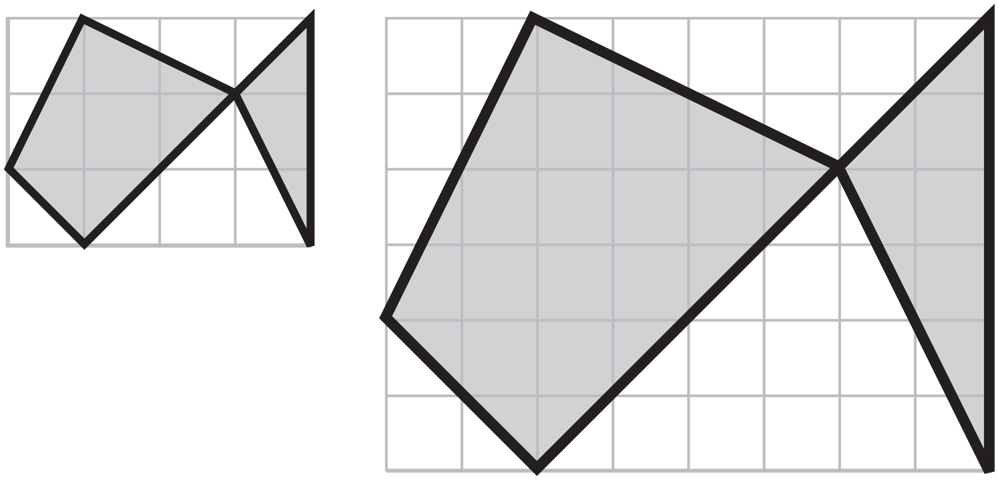
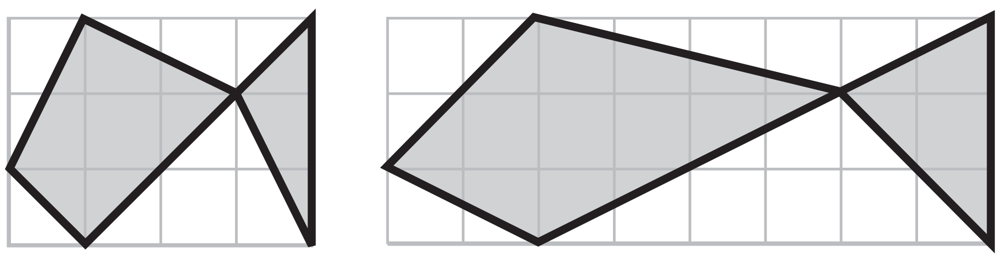
Line Master 10 **Changing Shape**





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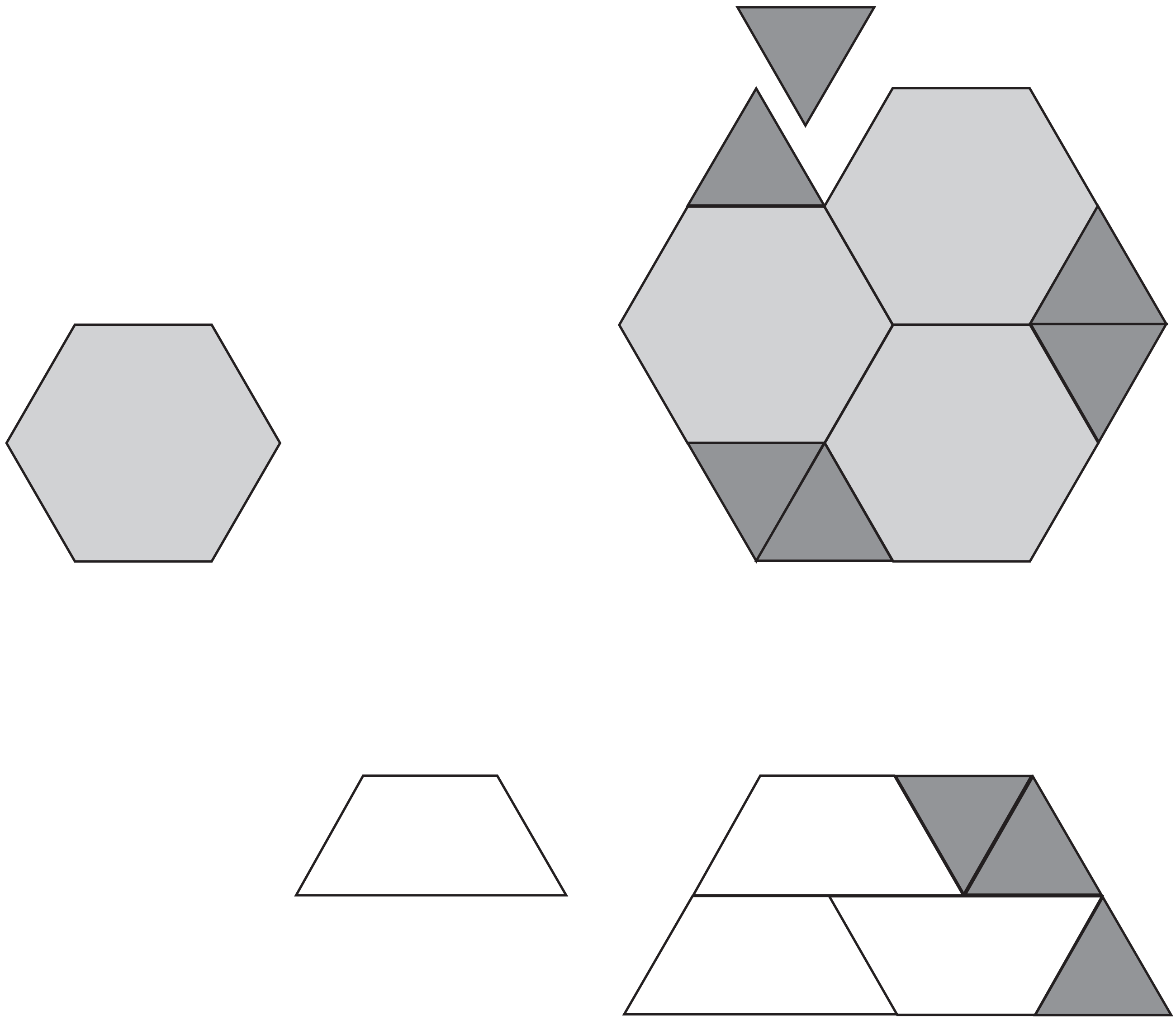
Line Master 11 **Hexagons and Trapezoids**



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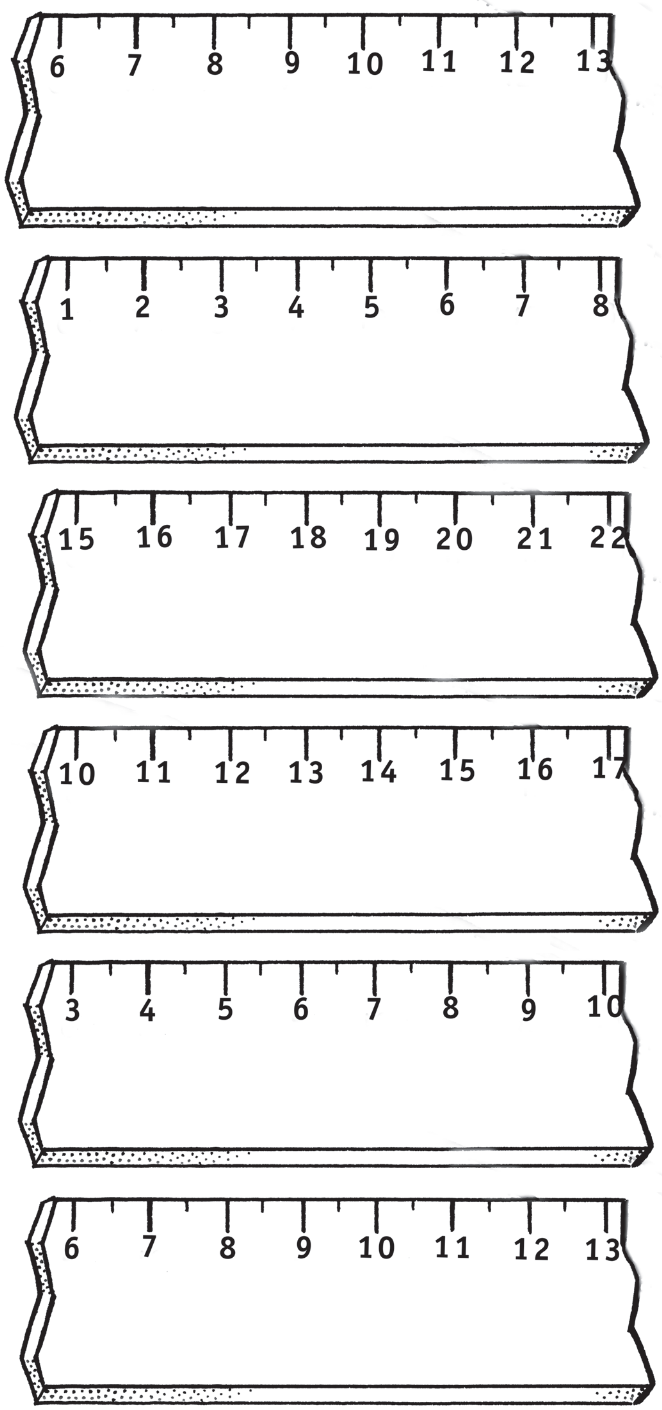
Line Master 12 **Broken Rulers**



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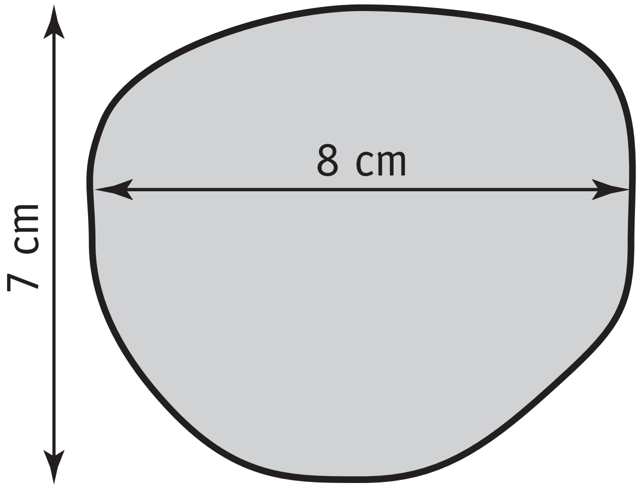
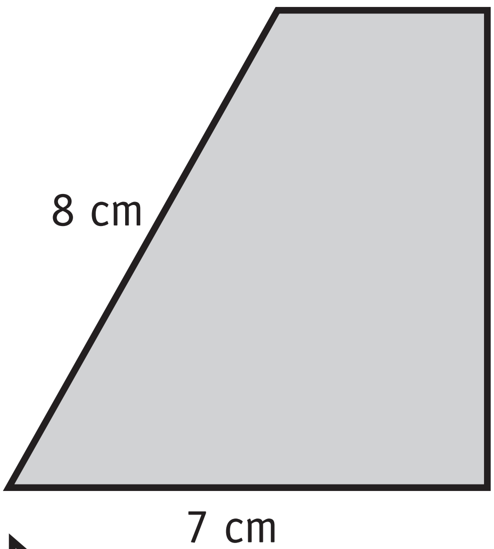
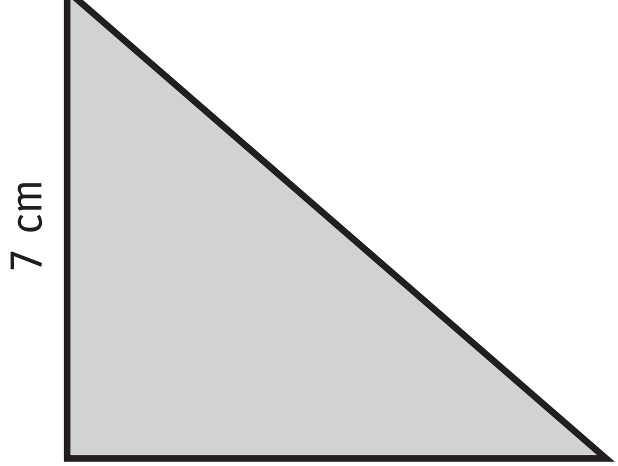
Line Master 13 **Using a Formula**







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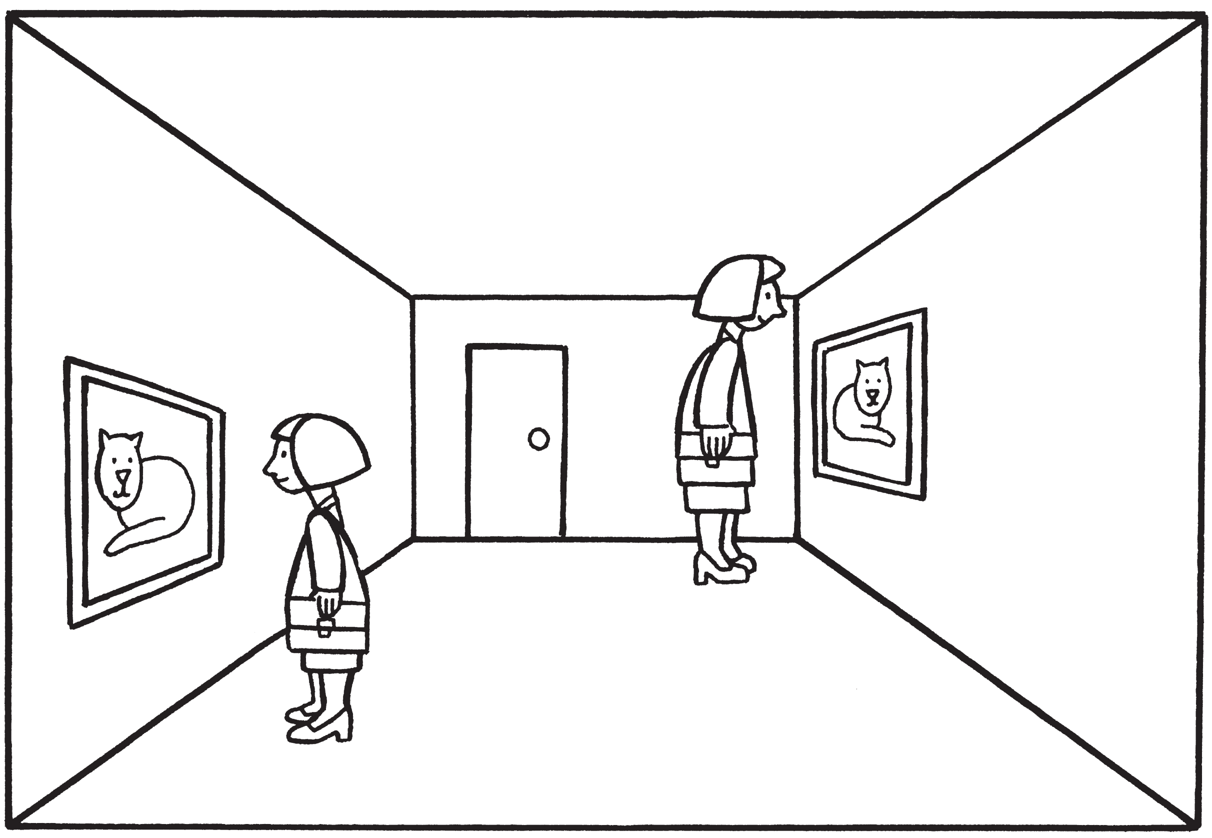
Line Master 14 **Distorted Estimates**



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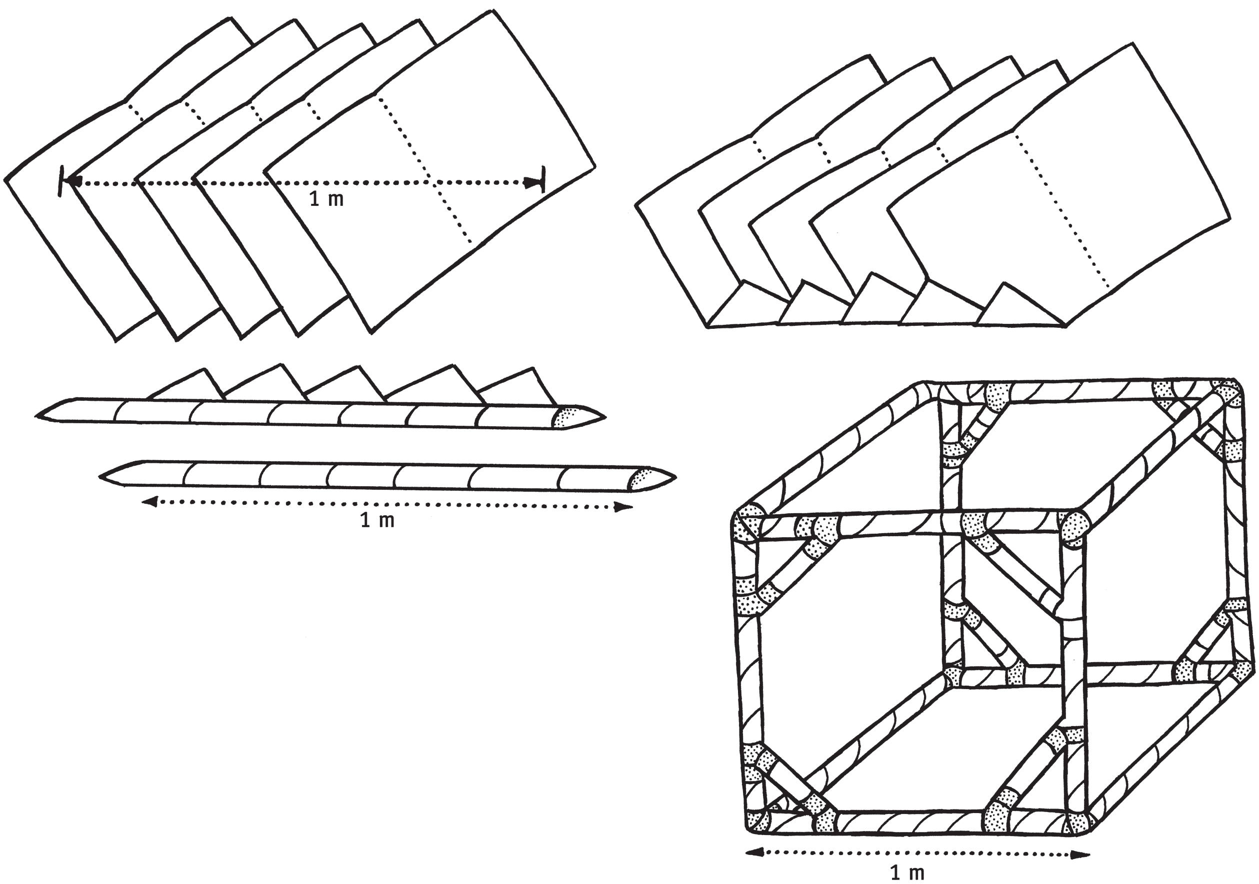
Line Master 15 **Volume of the Classroom**



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Line Master 16 **1-cm Grid Paper**

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Line Master 17 **2-mm Grid Paper**

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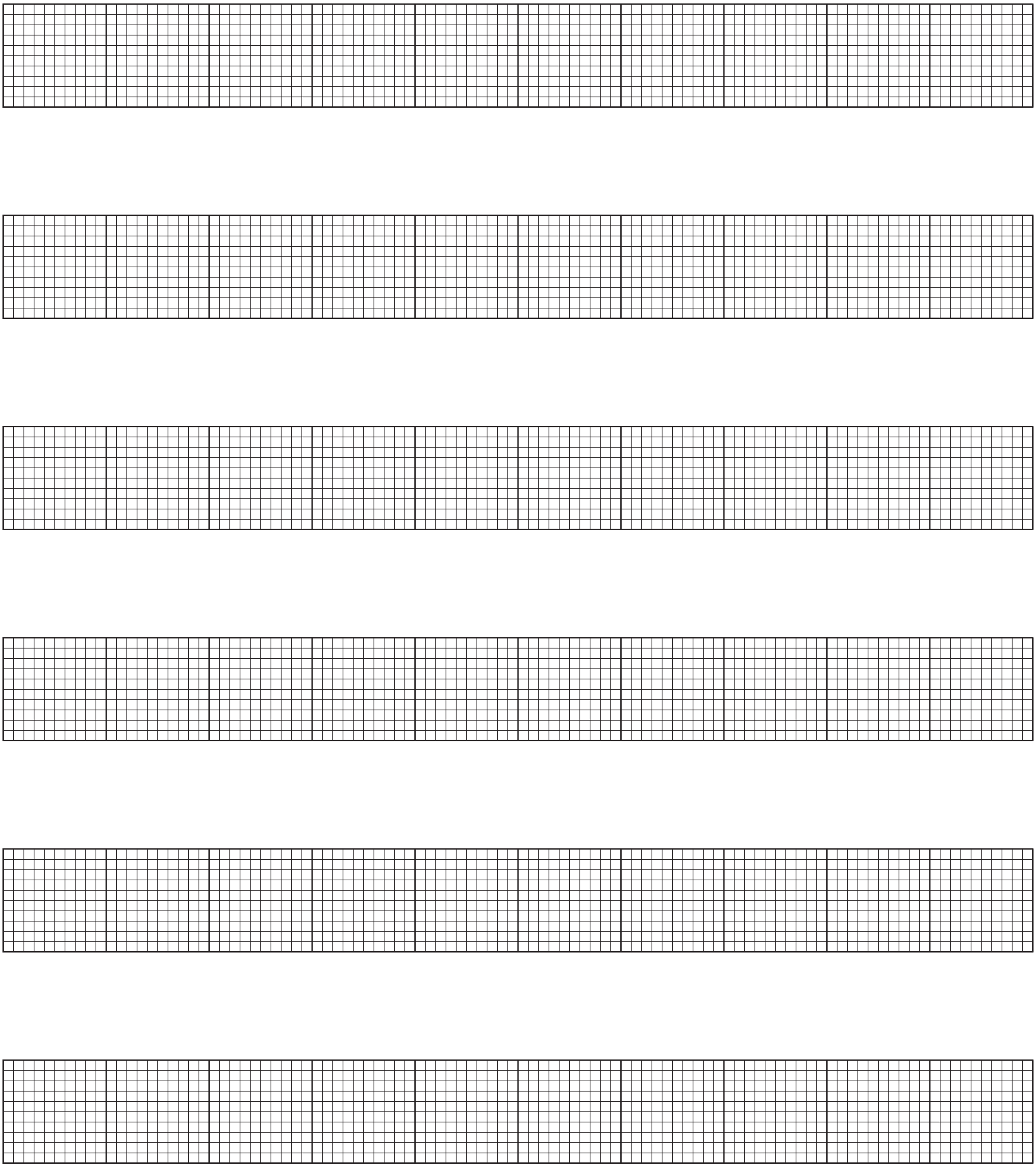
Line Master 18 **1000 Grids**



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Name: ____________________ Date: ____________________

Line Master 19 **Ten-Squared Grid Paper**

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Name: ____________________ Date: ____________

Line Master 20 **10 x 10 Array**

Name: ____________________ Date: ____________________

Line Master 21 **1-mm Grid Paper**

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Planning Master

Classroom Plan for Week ________, Term ________ Grade Level: ____________

Curricular Goal/ Key Understanding	Mathematical Focus	Activities	Focus Questions	Observations/ Anecdotes

Emergent Phase

During the Emergent Phase

Students initially attend to overall appearance of size, recognizing one thing as perceptually bigger than another and using comparative language in a fairly undifferentiated and absolute way (big/small) rather than to describe comparative size (bigger/smaller). Over time, they note that their communities distinguish between different forms of bigness (or size) and make relative judgements of size.

As a result, they begin to understand and use the everyday language of attributes and comparison used within their home and school environment, differentiating between attributes that are obviously perceptually different.

By the end of the Emergent phase, students typically:

- distinguish tallness, heaviness, fatness, and how much things hold
- start to distinguish different forms of length and to use common contextual length distinctions; e.g., distinguish wide from tall
- use different bipolar pairs to describe things; e.g., thin–fat, heavy–light, tall–short
- describe two or three obvious measurement attributes of the same thing; e.g., tall, thin, and heavy
- describe something as having more or less of an attribute than something else, e.g., as being taller than or as being fatter than

These students recognize that numbers may be used to signify quantity.

Matching and Comparing Phase

Most students will enter the Matching and Comparing phase between 5 and 7 years of age.

As students move from the Emergent phase to the Matching and Comparing phase, they:

- may not "conserve" measures; e.g., thinking that moving a rod changes its length, pouring changes "how much," cutting up paper makes more surface
- may visually compare the size of two things, but make no effort to match; e.g., saying which stick is longer without lining up the bases or which sheet of paper is bigger without superimposing
- compare time spans but may not take into account different starting times; e.g., deciding that the TV program that finished latest was on longest
- use bipolar pairs but may have difficulty with some comparative terms; e.g., lift to decide which is heavier but say both are heavy because both hands go down
- may distinguish two attributes (such as tallness and mass) but not understand that the two attributes may lead to different orders of size for a collection, expecting the order for tallness and the order for mass to be the same
- while describing different attributes of the same thing (tall, thin, and heavy) may be confused by a request to compare two things by different attributes, particularly if the comparisons lead to different orders
- often do not think to use counting to say how big or how much bigger; e.g., they may "weigh" something by putting it into one side of a balance and smaller objects into the other side but not count the objects

During the Matching and Comparing Phase

Students match in a conscious way in order to decide which is bigger by familiar, readily perceived, and distinguished attributes such as length, mass, capacity, and time. They also repeat copies of objects, amounts, and actions to decide how many fit (balance or match) a provided object or event.

As a result, they learn to directly compare things to decide which is longer, fatter, heavier, holds more, or took longer. They also learn what people expect them to do in response to questions such as "How long (tall, wide or heavy, much time, much does it hold)?" or when explicitly asked to measure something.

By the end of the Matching and Comparing phase, students typically:

- attempt to focus on a particular attribute to compare two things; e.g., how much the jar holds
- know that several things may be in different orders when compared by different attributes
- line up the base of two sticks when comparing their lengths and fit regions on top of each other to compare area
- use the everyday notion of "how many fit" and count how many repeats of an object fit into or match another; e.g., How many pens fit along the table? How many potato prints cover the sheet? How many blocks fit in the box?
- count units and call it "measuring;" e.g., *I measured and found the jar holds a bit more than 7 scoops*.
- refer informally to part-units when measuring uni-dimensional quantities; e.g., *Our room is 6 and a bit metres long*.

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Quantifying Phase

Most students will enter the Quantifying phase between 7 and 9 years of age.

As students move from the Matching and Comparing phase to the Quantifying phase, they:

- while knowing that ordering objects by different attributes may lead to different orders, may still be influenced by the more dominant perceptual features; e.g., they may still think the tallest container holds the most
- may count "units" in order to compare two things but be fairly casual in their repetition of units, not noticing gaps or overlaps; e.g., placing the first "unit" away from the end when measuring length, not worrying about spills when measuring how much a container holds, not stopping their claps immediately the music stops
- do not necessarily expect the same "answer" each time when deciding how many fit
- may not think to use unit information to answer questions such as: Which cup holds more? Will the table slide through the door?
- may not see the significance of using a common unit to compare two things and, when using different units, let the resulting number override their perceptual judgement
- while many will have learned to use the centimetre marks on a conventional rule to "measure" lengths, they often do not see the connection between the process and the repetition of units

During the Quantifying Phase

Students connect the two ideas of directly comparing the size of things and of deciding "how many fit" and so come to an understanding that the count of actual or imagined repetitions of units gives an indication of size and enables two things to be compared without directly matching them.

As a result, they trust information about repetitions of units as an indicator of size and are prepared to use this in making comparisons of objects.

By the end of the Quantifying phase, students typically:

- attempt to ensure uniformity of representations of the unit; e.g., check that the cup is always full, the pencil does not change length, the balls are the same size
- use the representations of their unit carefully to make as close a match as possible, avoiding gaps and overlaps; e.g., choose a flexible tape to measure the perimeter of a curved shape
- know why they need to choose the same size objects to use as units when comparing two quantities
- see repeating one representation of the unit over and over as equivalent to filling or matching with multiple copies of it
- connect the repetition of a "unit" with the numbers on a whole-number calibrated scale
- make things to a specified length in uniform units (including centimetres and metres)
- use provided measurements to make a decision about comparative size; e.g., use the fact that a friend's frog weighs 7 marbles to decide whether their own frog is heavier or lighter
- count units as a strategy to solve comparison problems such as: Whose frog is heavier? Put the jars in order from the one that holds the most to the one that holds the least.
- are prepared to say which is longer (heavier) based on information about the number of units matching each object
- think of different things having the same "size;" e.g., use grid paper to draw different shapes with the same perimeter
- add measurements that they can readily think of in terms of repetitions of units; e.g., find the perimeter of a shape by measuring the sides and adding

Measuring Phase

Most students will enter the Relating phase between 11 and 13 years of age.

As students move from the Quantifying phase to the Measuring phase, they:

- while trying to make as close a match as possible to the thing to be measured, may find the desire to match closely overriding the need for consistency of unit; e.g., they may resort to "filling" a region with a variety of different objects in order to cover it as closely as possible
- may not understand that the significance of having no gaps and overlaps is that the "true" measurement is independent of the placement of the units
- may still think of the unit as an object and of measuring as "fitting" in the social sense of the word (How many people fit in the elevator? How many beans in the jar?) and so have difficulty with the idea of combining part-units as is often needed in order to find the area of a region
- maNy confuse the unit (a quantity) with the instrument (or object) used to represent it; e.g., they may think a square metre has to be a square with sides of 1 metre, may count cubes for area and not think of the face of each as the unit
- may interpret whole numbered marks on a calibrated scale as units but may not interpret the meaning of unlabelled graduations

During the Measuring Phase

Students come to understand the unit as an amount (rather than an object or a mark on a scale) and to see the process of matching a unit with an object as equivalent to subdividing the object into bits of the same size as the unit and counting the bits.

As a result, they see that part-units can be combined to form whole units and they understand and trust the measurement as a property or description of the object being measured that does not change as a result of the choice or placement of units.

By the end of the Measuring phase, students typically:

- expect the same number of copies of the representation of their unit to match the object being measured regardless of how they arrange or place the copies
- understand that the smaller the unit the greater the number; e.g., are able to say which is the longer of a 1-km walk and a 1400-m walk.
- compose "part-units" into wholes, understanding, for example, that a narrow garden bed may have an area of 5 or 6 m^2 even though no whole "metre squares" fit into the bed
- can themselves partition a rectangle into appropriate squares and use the array structure to work out how many squares are in the rectangle
- interpret the unnumbered graduations on a familiar whole-number scale
- understand the relationship between "part-units" and the common metric prefixes; e.g., know that a unit can be broken into one hundred parts and each part will be a centi-unit
- work with provided measurement information alone; e.g., order measurements of capacity provided in different standard units, make things which meet measurement specifications

Relating Phase

Most students will enter the Measuring phase between 9 and 11 years of age.

As students move from the Measuring phase to the Relating phase, they:

- while partitioning a rectangle into appropriate squares and using the array structure to find its area, may not connect this with multiplying the lengths of the sides of a rectangle to find its area
- while understanding the inverse relationship between the unit and the number of units needed, may still be distracted by the numbers in measurements and ignore the units; e.g., say that 350 g is more than 2 kg
- while converting between known standard units, may treat related metric measures just as they would any other unit, not seeing the significance of the decimal structure built into all metric measures

During the Relating Phase

Students come to trust measurement information even when it is about things they cannot see or handle and to understand measurement relationships, both those between attributes and those between units.

As a result, they work with measurement information itself and can use measurements to compare things, including those they have not directly experienced, and to indirectly measure things.

By the end of the Relating phase, students typically:

- understand that known relationships between attributes can be used to find measurements that cannot be found directly; e.g., understand that we can use length measurements to work out area
- know that for figures of the same shape (that is, similar) the greater the length measures the greater the area measures, but this is not so if the figures are different shapes
- understand why the area of a rectangle and the volume of a rectangular prism can be found by multiplying its length dimensions and can use this for fractional side lengths
- think of the part-units themselves as units; e.g., a particular unit can be divided into one hundred parts and each part is then a centi-unit
- subdivide units to make measurements more accurate
- choose units that are sufficiently small (that is, accurate) to make the needed comparisons
- use their understanding of the multiplicative structure built into the metric system to move flexibly between related standard units; e.g., they interpret the 0.2 kg mark on a scale as 200 g
- notice and reject unrealistic estimates and measurements, including of things they have not actually seen or experienced
- use relationships between measurements to find measures indirectly; e.g., knowing that 1 mL = 1 cm^3 they can find the volume of an irregular solid in cubic centimetres by finding how many millilitres of water it displaces using a capacity cylinder

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