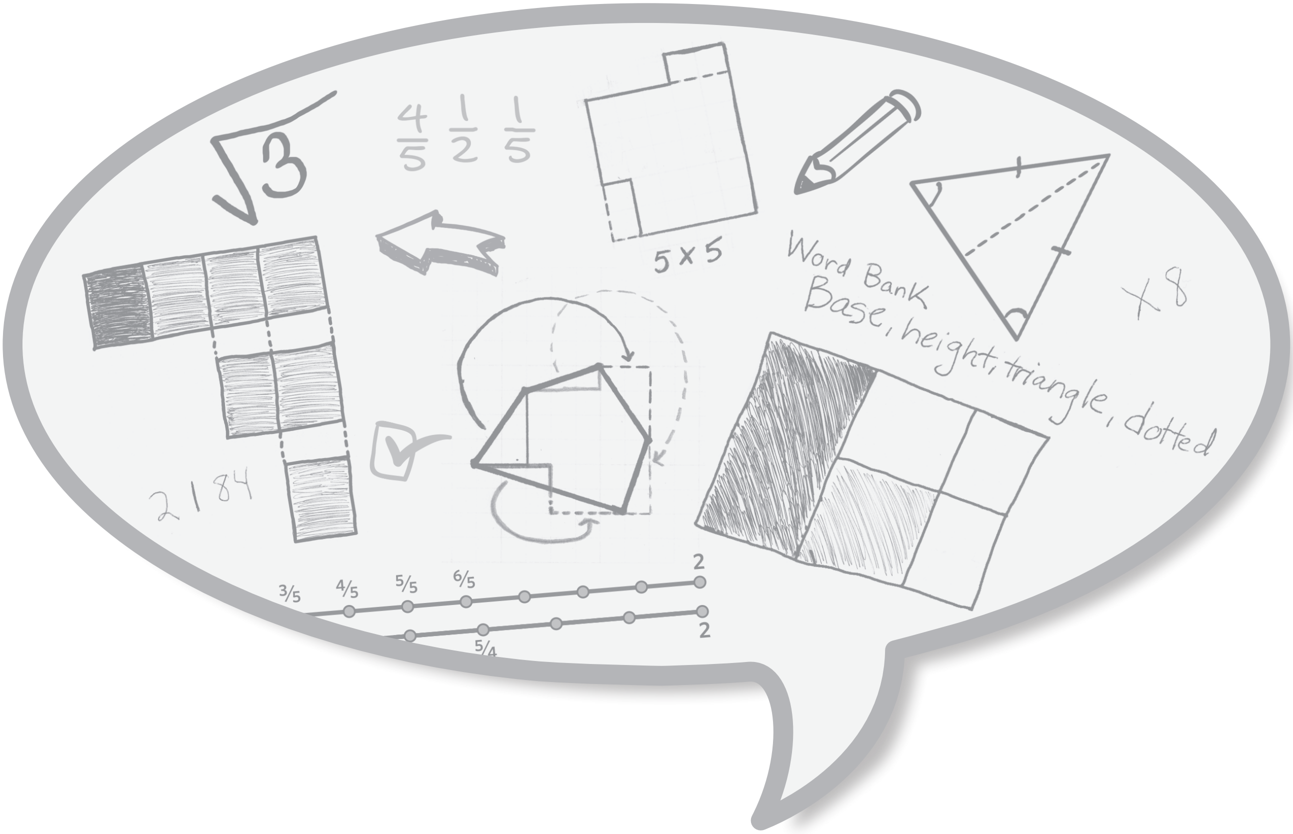

Mathematical Thinking *and* Communication

Access for English Learners

INCLUDES **ONLINE** PD RESOURCES

MARK DRISCOLL
JOHANNAH NIKULA
JILL NEUMAYER DEPIPER

Heinemann
Portsmouth, NH

Heinemann
361 Hanover Street
Portsmouth, NH 03801–3912
www.heinemann.com

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Introduction

> "Words create worlds, he used to tell me when I was a child. They must be used very carefully. Some words, once having been uttered, gain eternity and can never be withdrawn."
>
> —Susannah Heschel, daughter of Rabbi Abraham Heschel[1]

Laments about the state of American schools seem to occur frequently in the popular press. District, state, and federal mandates often follow. And when the articles and mandates turn to potential solutions, these solutions often sound like zero-sum, or win-lose, propositions, with underlying messages that can be heard, particularly by teachers, as, "In order for all students to make gains, teachers must do more." From our work with teachers, coaches, and schools, we believe that better solutions are those that are win-win, with teachers doing what they signed up for in the first place when they entered the teaching profession. In particular, we want teachers to gain as students gain, with "better" solutions arising from teachers doing familiar things perhaps *differently*, rather than "better" solutions being equated with teachers layering *more* onto what they have been doing.

The key, we believe, is to remember that students, particularly English learners (ELs), are *thinking* when they engage in mathematics tasks and, moreover, using language as they think. What if that thinking could be made visible and audible and caused to blossom into productive

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[1] A. J. Heschel and S. Heschel, *Moral Grandeur and Spiritual Audacity* (New York: Farrar, Straus, and Giroux, 1996), ix–x.

mathematical practices and mathematical communication? That would benefit students while benefiting teachers. Right there would be a win-win solution, and it would emanate from mathematics tasks already used or tasks similar to those already used. We argue in this book that familiar tasks can be enhanced for ELs with listening/reading supports and speaking/writing prompts, along with prompts for ELs to use mathematical visual representations to propel and communicate their thinking.

To move from the status quo to the win-win solution, mathematics teachers of ELs have particular challenges to overcome, especially two commonly held opinions. First, there is a widespread belief that students need English proficiency in order to do mathematics reasoning tasks. A related conviction holds that the best way to create access to mathematics tasks for ELs is to lighten the cognitive demand of the tasks. Neither the "English first" nor the "lighten the cognitive demand" strategies about mathematics tasks for English learners is necessary for building proficiency in mathematical reasoning. Instead, we suggest remembering that ELs are thinking when working on mathematics tasks, and by letting that thinking become more visible and audible, teachers can nurture the productive potential in that thinking, as well as help adjust any faulty or misinformed thinking.

This book is intended to be a resource for mathematics teachers whose students include ELs. Furthermore, because this book explores the roles that language plays in the learning of mathematics, we also believe it can be useful to all teachers of mathematics, regardless of the particular collection of students in their classrooms. But these strategies, tailored in ways to meet the needs of different students, are absolutely essential for ELs. To meet ELs' needs it is necessary to nurture mathematics teaching practices that "specifically address the language demands of students who are developing skill in listening, speaking, reading, and writing in a second language while learning mathematics" (Celedón-Pattichis and Ramirez 2012, 1). In other words, although ELs must gain facility in using English to express themselves mathematically, in order to succeed in mathematics, the learning can and should happen "while learning mathematics."

When we talk about ELs, we mean students for whom English is not their home or first language and whose current English language proficiency level potentially interferes with their grade-level mathematics work. ELs, like other students, are by no means all the same, and each brings different strengths and struggles to the classroom. We recognize the importance of attending to (1) students' different English-language proficiency levels, cultural backgrounds, schooling backgrounds, first languages, current mathematical understanding, and so on, and (2) classrooms' different makeups in regard to the mix of EL and non-EL students as well as the background of EL students in the class. We intend this book to be useful for all teachers in thinking about supporting the ELs they happen to teach.

Background to Our Work

This book grows out of a decade of work with mathematics teachers of ELs, in a wide range of districts—including large urban, small urban, suburban, and rural. The first languages of ELs in these districts were both numerous and varied, as were the policies created to serve their needs. That set of experiences allowed us to hone a set of ideas and strategies to increase access to mathematics learning opportunities in English-speaking classrooms. The ideas and strategies were mainly tested in middle-grade classrooms, but we believe they can be adapted to both lower and higher grades.

Our interest in working with mathematics teachers of ELs began a decade ago in New York City. We were asked to conduct a seminar series, with an emphasis on analyzing student work on challenging mathematics tasks. The invited school teams were focused on improving the mathematics performance of English learners in their middle schools. Each team included an ESL specialist, so the teams were well advised in the English-as-second-language needs of EL students. We were not asked there because they thought us knowledgeable about those needs. While relieved about that point, our team still felt a bit at sea and disconnected from participant needs, since we had so little experience working with ELs at that point.

This uneasiness dissolved quickly for us in the early weeks of the project, when the director of the New York City Office of English Language Learners came to a seminar and addressed the participating school teams with words to the effect of:

> For English learners to succeed in learning mathematics, they need to be more *productive* in mathematics classrooms—reasoning more, speaking more, writing more, drawing more.

For our team leading the seminars, this statement had a liberating effect by enabling us to recognize that, as people with experience helping others reason more, speak more, write more, and draw more in mathematics, we did have much to contribute to the efforts of the school teams. This was a major mind-set shift.

In that vein, and briefly put, this book is designed to make a case for mathematics teachers and coaches to secure a similar mind-set. We want teachers and coaches to recognize that their accumulated knowledge and skill in helping students be more productive during mathematics lessons also apply to meeting the learning needs of EL students, albeit, with some targeted shifts in strategy, so that language supports and visual representations can play salient roles. Of course, such a belief would do no good unless put into practice. And so, the book offers ways in which we believe teachers can enrich their current practice to create access for EL students to proficiency in mathematical reasoning and development of mathematical practices.

Guiding Perspectives

We outline below four perspectives that have guided our own work and the work of our collaborating teachers. We hope they will be useful to you in your own efforts to improve access to proficiency in mathematical reasoning and development of mathematical practices among ELs. Underlying the perspectives is an assumption about teaching and teacher learning: We assume that the vast majority of teachers would very much like to devote time to activities that provide them enjoyable and useful learning, including new ways to think about mathematics, about language, and about relationships between mathematics and language. These activities might plumb the depths of potential in mathematical visual representations; provide ideas and strategies useful to engage hard-to-reach students; and invite collaborative, professional problem solving. At the same time, we recognize that many other, often-mandated, activities can make time scarce for our suggested professional development. Therefore, we offer in this book ideas, activities, and strategies in the hopes that they can be employed when windows of time do open up.

The guiding perspectives are that we mathematics educators should:

1. **Adopt useful, actionable definitions of equity.** Defining equity can be an elusive task. It cannot mean "equal treatment for all," because that could never be achieved in a world where material and financial resources are distributed so unevenly. Thus, that definition just does not seem practical. Another definition candidate, "A fair chance for everyone," appears to pass the practical test, but the word *fair* requires unpacking. In our experience, *access* is a key piece of fairness, that is, providing each learner alternative ways to achieve, no matter the particular obstacles he or she faces. A related piece is *potential*, as in the

potential shown by students to do challenging mathematical reasoning and problem solving. Seeing fairness in terms of access and potential can give more concrete meaning to "fairness for all" as groundwork for equity.

These words all prompt wondering—in the case of *access*, about what is currently missing, and in the case of *potential*, about what might be done next to create access. They invite problem solving to find ways around obstacles in the path of student growth. For instance, what stands in the way of ELs' becoming proficient in solving mathematical word problems? To a large extent, the words in the problems constitute the obstacle. One alternate access route could involve visual representations, especially if they are complemented by a set of carefully matched language strategies. Similarly, when ELs or other students struggle to solve a mathematical task and show evidence of potentially proficient thinking, what stands in the way of their developing robust mathematical practices, as described in the Common Core Standards of Mathematical Practice? One likely stumbling point is a curriculum that is overloaded with computational procedures. With this in mind, we have advocated students to engage with geometric reasoning tasks as an access route toward fulfilling potential in mathematical practices.

2. **Take an expansive view of the role of language in mathematics and in teaching mathematics.** In his book, *The Language of Mathematics*,[2] Barton writes, "Mathematical concepts, objects, and relationships arise through language, and within particular socio-cultural environments, in response to human thinking about quantity, relationships, and space" (88). Given this intimate relationship between mathematics and language, and given the pervasive importance of quantity, relationships, and space in mathematics, it makes sense that all teachers, but especially teachers of English learners, would want to develop habits of heightened attention to language. An example related to this point comes from our work on fostering geometric thinking,[3] where we videotaped three eighth graders working on one of our geometric dissection problems. In the problem, the students were asked to come up

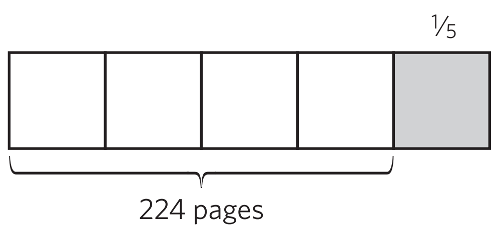
[2] Bill Barton, *The Language of Mathematics: Telling Mathematical Tales* (New York: Springer Science+Business Media, 2009).

[3] M. Driscoll, R. Wing DiMatteo, J. Nikula, and M. Egan, *Fostering Geometric Thinking* (Portsmouth, NH: Heinemann, 2007).

with a method to dissect a provided parallelogram and move the pieces to make a rectangle. The last question was: "Will your method work for any parallelogram?" One of the students drew a parallelogram, tried the method, and declared, "It works!" A second student objected, "But there are other parallelograms," to which the first student asked, "*What* other parallelograms?" The other student fell silent, apparently not knowing how to respond.

After several viewings of this segment, we concluded that the student who objected took "any" to mean what was intended—namely, a synonym for "every." The first student, on the other hand, seemed to interpret "any" in the way it is used in a sentence, like "Do you have any opinion about where to go for lunch?" In this case, one opinion suffices as a response. In mathematics, however, "any" usually has a privileged meaning: every. There are many such instances of words and phrases with privileged meanings in mathematics, and so it behooves all of us to be alert to the use of language in mathematical matters.

3. **Take an expanded view of mathematical proficiency, emphasizing the quality of mathematical thinking.** To some, the advent of the Common Core Standards of Mathematical Practice has presented yet another hurdle for English learners on the road to mathematics proficiency. We have come to recognize, however, that one can look at the eight Standards of Mathematical Practice (SMP) as a lens on EL student potential for mathematical growth and as pointers for teachers toward providing access for EL students to mathematical growth. In Chapter 3 of this book, we discuss our use of a particular problem in our research with English learners: *Rita has read 224 pages of her book. She has $^1/_5$ of the book left to read. What is the total number of pages in the book?* Several of the students drew a diagram like this:



This looks very promising, pointing toward possible next steps of

1. dividing 224 by 4 to get 56
2. recognizing that 56 pages are in each of the four white parts, so
3. there also must be 56 pages in the shaded part. Hence,
4. the total number of pages is 224 + 56 = 280 pages.

These students did none of this, just leaving the diagram they'd constructed. They appeared stuck.

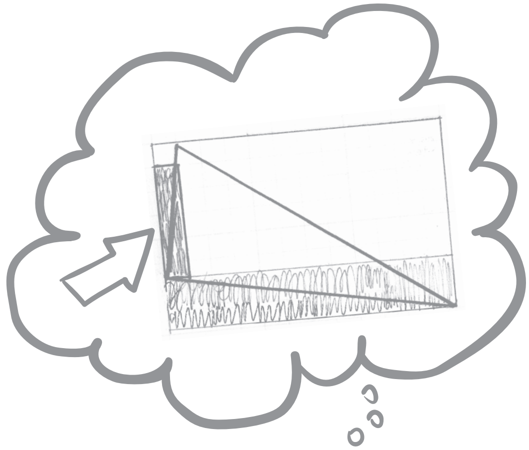
Although they may have been several steps from solution, their record of thinking shows strong potential related to SMP 2: *Reason abstractly and quantitatively*, because they have portrayed diagrammatically the relationship between the quantity "pages read" and the quantity "pages left to read." Furthermore, there is an implicit representation of the relationship between the numbers $^1/_5$ and $^4/_5$. A teacher might seize on this evidence of potential to ask questions to advance the student thinking, such as "In your diagram, why are there four parts for 224 pages?" "How many pages in each of the four parts?"

When measured by the gauge "Did they get it?" the answer is sometimes no, but often students' work reveals evidence of SMP potential. We can use that evidence to move the students forward by recognizing it and asking helpful questions to advance thinking.

4. **Keep evidence of student mathematical thinking and communication at the center of focus, and use mathematics tasks that allow this focus.** Student and teacher learning opportunities will thrive when mathematical tasks are employed that elicit student thinking and communication and create records of student thinking and communication, and when teachers' planning around tasks is informed by attention to use of language. In this book, we reflect a strong belief we gained from our work and from reading the research literature in mathematics education: All students benefit from a steady diet of tasks that elicit mathematical reasoning. These tasks can prompt problem solving, analytical reasoning, spatial/geometric reasoning, or quantitative reasoning. Chapter 1 provides examples of each.

The material in this book reflects these perspectives, and we hope that it supports you in adopting them as well. Chapter 1 looks very closely at the topic of access for English learners, with particular attention on access to several ways of reasoning in mathematics; Chapter 2 focuses on the roles mathematical visual representations play in providing ELs access to mathematical thinking and communication; Chapter 3 describes the rationale for teacher analysis of ELs' visual representations as a means to tap EL student potential, along with a framework for analysis; Chapter 4 looks closely at what is meant by learning to communicate mathematically while doing and learning mathematics; Chapter 5 considers the qualities that make tasks conducive to expanding access for EL students; Chapter 6 describes instructional routines that can embed tasks in lessons so that EL access is maximized; and finally, Chapter 7 summarizes major ideas in the book and contains suggestions for ways to start implementing the ideas in your practice.

Chapter 1



Creating Access to Mathematics for English Learners

The notion of *access* has been an important beacon for us in our work alongside teachers of students who are English learners. ELs may face many challenges that can impede success in learning mathematics. After we identify these challenges, we seek avenues of access so students can surmount the challenges, building on their strengths as often as possible. Access to the academic English used in instruction, textbooks, and tests is essential, of course, but so is access to opportunities to solve problems and reason mathematically even before students gain high levels of English proficiency.

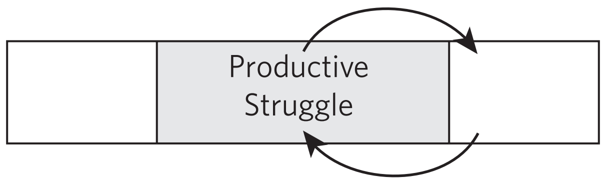
How Language Is Involved in Mathematical Reasoning

Our efforts to broaden access proceed along the following line of reasoning:

- Language is deeply involved in the learning of mathematics. At the same time,

- English proficiency is *not* a prerequisite for doing mathematical work that is cognitively demanding, such as reasoning and problem solving. That is because
- Mathematical visual representations and other thinking tools and language tools can be employed to support the integration of academic English into mathematical reasoning and problem solving.

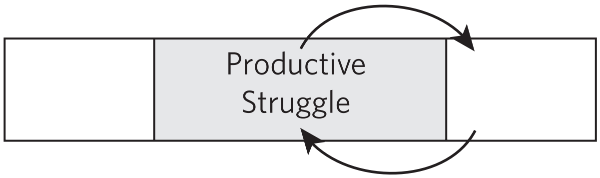
This line of reasoning may challenge beliefs and defy expectations for some and so deserves further unpacking. Suppose a mathematics problem is presented to students. Ideally, most of the students' efforts will include productive struggle with the mathematics, with some beginning "stuff" up front and some closing "stuff" at the end. The productive mathematical struggle in the middle is, for the most part, internal, silent, and driven by previous experience, so it usually doesn't require knowledge of English.



However, the beginning "stuff" (the white rectangle on the left) is the entry door into the task, requiring students' efforts to *make meaning* of the presented task; meanwhile, the "stuff" at the end (the white rectangle on the right) consists of students *communicating* how they reasoned about the task. Students may go back and forth between productive struggle about the mathematics and communicating how they reasoned about that mathematics, thus the cyclical arrows. Both the beginning work to gain access to the task and the later work to communicate based on the productive struggle with the mathematics are wedded to use of written and spoken language.

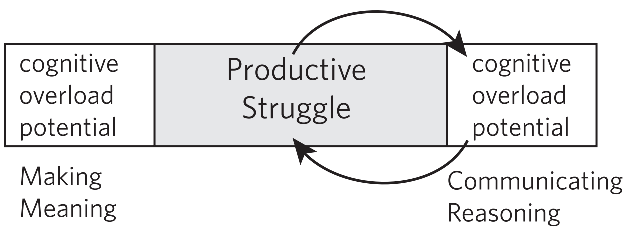
For example, imagine that the students are presented with a written word problem, and suppose further that the teacher has asked students to cap their mathematical work on the problem by explaining to a partner how they reasoned and solved the problem.

In this case, the challenges in the beginning stuff might be confusing words and phrases in the problem statement or general difficulty tracking the narrative in the problem statement, perhaps because of an unfamiliar context. Also, ELs (like other students) may be looking for cues that tell them to add, subtract, multiply, or divide. This is usually a waste of attention, since words like *more* may signify the need to add but could also involve multiplication or even subtraction. Challenges in the right side of the rectangle may include anxiety about writing an explanation—how to use precise language and full sentences, for example. Also challenging can be the task of translating words in the problem to symbols used in thinking about solving it. Finally, ELs may be especially challenged in attaching words to how they were thinking and what decisions they made along the way.



Cognitive Load and Cognitive Overload

Clearly, this "stuff" before and after the mathematical thinking can trip up ELs, often pushing them into what is called "cognitive overload." The concept of *cognitive load* refers to work the brain does when processing information, within the limits of working memory. Research shows that working memory is limited in how much information it can hold at any one time.[1] Hence, cognitive *overload* can happen quickly—with working memory buckling under too much information to process in too short a time. Returning to the word-problem example: All students, to varying degrees, struggle to do the mathematical work required by problems; however, working memory in ELs is strained even further by language demands, in struggles to draw meaning from problem statements, as well as later in communicating reasoning to others.



[1] J. Sweller, "Cognitive Load During Problem Solving: Effects on Learning," *Cognitive Science* 12 (1988): 257–85.

Supports to Reduce Cognitive Load

On the basis of our work with schools, we are convinced that lack of English language proficiency need not preclude ELs from productive struggle with challenging mathematics tasks (involving reasoning and problem solving), *provided* appropriate thinking and language tools are used to support ELs in gaining access to the task and in communicating their thinking about the task—i.e., the "stuff" that comes before and after productive struggle. A key aspect of "appropriate" that we emphasize is that the tools are woven seamlessly into students' implementation of the mathematics tasks. Furthermore, we believe that all students—not just ELs—can benefit from using them. Appropriate tools include mathematical diagrams for word problems, enhanced geometric drawings for geometric reasoning tasks, language access strategies for analyzing the meaning in the wording of tasks, and language production strategies for helping students communicate their mathematical thinking. Examples appear throughout the rest of this chapter and the book.

Types of Mathematical Reasoning

Our work has involved using tasks to help students—especially ELs—engage with different kinds of mathematical productive struggle. We aim to promote both mathematical problem solving and analytic reasoning, and we focus on contexts that allow spatial/geometric reasoning and quantitative reasoning. These four categories are by no means mutually exclusive. In fact, you will see examples of tasks and student work in this book for which you might rightly say, "That seems like a combination of problem solving and geometric reasoning," or "On that task, students seemed to be using analytic reasoning and quantitative reasoning to support their problem solving." Before we show examples of each of the categories, here are brief definitions.

Problem Solving

We think of mathematical problem solving as an umbrella that involves careful strategizing and can include the other kinds of mathematical reasoning we describe below. The 2012 What Works Clearinghouse practice guide publication, *Improving Mathematical Problem Solving in Grades 4 Through 8*, says that mathematical problem solving "involves reasoning and analysis, argument construction, and the development of innovative strategies" (6).

The guide lays out research-based arguments and evidence regarding access for all students to proficiency in mathematical problem solving. Among the principal recommendations was regular classroom use of visual representations, such as diagrams: "Students who learn to visually represent the mathematical information in problems prior to writing an equation are more effective at problem solving" (Woodward et al. 2012, 23).[2] This, of course, is a recommendation for all students, but teachers of ELs should see the additional benefits for non-English speakers. In creating and analyzing diagrams and in manipulating geometric drawings, ELs can propel their mathematical thinking, as well as *reveal* their mathematical thinking to teachers, with minimal need to understand English. Furthermore, for mathematical word problems, diagrams can act as a bridge between the words of the problem and the symbolic calculations needed to determine a solution. In the words of one of our collaborating teachers, describing ELs' use of diagrams: "It is just worth everything because it gives them some way to access it and some way to get success."

Analytic Reasoning

Broadly speaking, analytic reasoning "refers to a set of processes for identifying the causes of events" (Siegler 2003).[3] As a form of mathematical thinking, it involves "distinguishing between features that typically accompany the use of a particular mathematical problem solving technique, and features that are essential for the technique to apply, (and) usually requires analysis of why the technique is appropriate or inappropriate" (229).

Analytic reasoning complements problem solving by broadening students' thinking about problem-solving strategies and by helping them

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[2] J. Woodward, S. Beckmann, M. Driscoll, M. Franke, P. Herzig, A. Jitendra, K. R. Koedinger, and P. Ogbuehi, *Improving Mathematical Problem Solving in Grades 4 Through 8: Practice Guide* (NCEE 2012-4055) (Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education, 2012). Retrieved from http://ies.ed.gov/ncee/wwc/publications_reviews.aspx#pubsearch/.

[3] R. S. Siegler, "Implications of Cognitive Science Research for Mathematics Education," in *A Research Companion to Principles and Standards for School Mathematics*, eds. J. Kilpatrick, W. B. Martin, and D. E. Schifter (Reston, VA: National Council of Teachers of Mathematics, 2003), 219–33.

transfer their problem-solving skills to unfamiliar situations, as described by Siegler:[4]

> Encouraging children to explain other people's reasoning in many contexts may lead children to internalize such an analytic stance to the point where they ask such questions reflexively, even when not prompted to do so. . . . When children are actively engaged in understanding why things work the way they do, transfer follows naturally and without great effort.

Spatial/Geometric Reasoning

By spatial/geometric reasoning, we mean reasoning about properties of geometric figures, reasoning about relationships between geometric figures, and reasoning about geometric measurement in figures, such as perimeter, area, and volume.

In 2008, we completed the Fostering Geometric Thinking project.[5] Two related discoveries in this research and development project were:

- Very many middle graders do not have opportunities to do spatial/geometric reasoning tasks.
- When students do get such opportunities, many (including many ELs) show themselves proficient.

Consequently, we believe spatial/geometric reasoning should have a prominent place in middle-grades classrooms. In his book *The Language of Mathematics*,[6] Barton nicely traces the interactions of language development and mathematical understanding, writing that "mathematical concepts, objects, and relationships arise through language, and within particular socio-cultural environments, in response to human thinking about quantity, relationships, and space" (88). This perspective led us to emphasize the language of quantities, relationships, and space in the shaping of our language strategies. We reason that this emphasis provides a very useful foundation to the full body of language used in mathematical communication and

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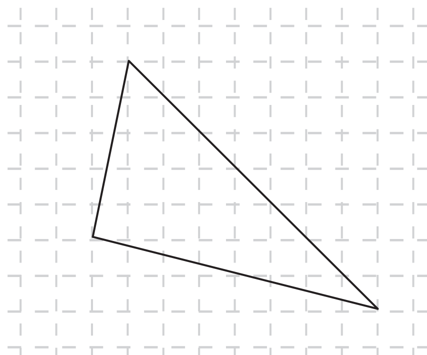
[4] R. S. Siegler, "Implications of Cognitive Science Research for Mathematics Education," 229.

[5] M. Driscoll, R. Wing DiMatteo, J. Nikula, M. Egan, J. Mark, and G. Kelemanik, *The Fostering Geometric Thinking Toolkit: A Guide for Staff Development* (Portsmouth, NH: Heinemann, 2008). M. Driscoll, R. Wing DiMatteo, J. Nikula, and M. Egan, *Fostering Geometric Thinking: A Guide for Teachers* (Portsmouth, NH: Heinemann, 2007). Both are products of Fostering Geometric Thinking in the Middle Grades Project (National Science Foundation ESI-0353409). Opinions expressed in this book are those of the authors and not necessarily the opinions of the National Science Foundation.

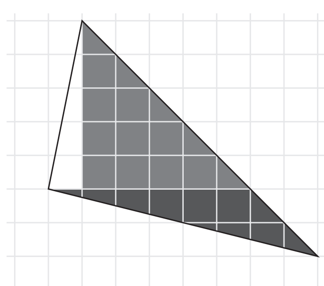
[6] B. Barton, *The Language of Mathematics: Telling Mathematical Tales* (New York: Springer Science+Business Media, 2009).

that mathematics teachers can make comfortable and profitable use of this foundation.

In the view of many American mathematics educators, including us, little reasoning is demanded in the geometry tasks that our middle school students see too frequently. For example, if "area" is involved in a task, students often seek out numbers they can multiply—as they believe they are supposed to do. "Area equals formula use" seems to be the unspoken assumption. This assumption works well in many tasks that involve area but not necessarily the areas of irregular figures like this triangle.



For this figure, finding the exact area requires considerable reasoning—e.g., thinking about the relationships between this triangle and the geometric figures defined by the underlying grid, perhaps leading to reasoning how one might use knowledge of areas of more regular figures (like a surrounding rectangle or composite interior triangles as shown here) to calculate the area of this irregular figure.



In our experience, without these reasoning options in their repertoires, many students make incorrect assumptions (such as, the angle on the lower left is a right angle), and they estimate side lengths (such as, the base side is

8 units and the height is 5 units). From there, they believe they can calculate area using the formula $^{bh}/_2$ (in this case, resulting in 20 square units, which is smaller than the actual area of 21 square units).

Quantitative Reasoning

From the Common Core Standards of Mathematical Practice (SMP) descriptions of mathematical practices, specifically Mathematical Practice 2, *Reason abstractly and quantitatively*: "Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them." In addition, quantitative reasoning involves "reasoning about the relationships among (quantities) without support of variable assignments or algebraic expressions" (99).[7]

Just as we want to emphasize spatial/geometric properties and relationships, as well as the language used to describe them, we also want to emphasize quantities and quantitative relationships and the language used to describe them. Much of early- and middle-grades mathematics is about quantities and relationships among quantities, but in our view, too often reasoning with and about quantities and relationships takes a back seat to applying computational procedures without much reasoning.

Another aspect of quantitative reasoning cited in the Common Core State Standards (CCSS) Mathematical Practices description is the ability to decontextualize and contextualize when using mathematics to solve problems. For example, suppose a word problem says that *Maria has \$10 more than Albert, and together they have \$40*. We can write a symbolic representation of this situation, say, M + A = (A + 10) + A = 40, with M and A representing, respectively, Maria's amount and Albert's amount. In doing so, we have *decontextualized* the situation, that is, abstracted the quantitative information from the situation. When we pause to check back in the problem, to see if we should be substituting A + 10 for M (as opposed, say, to something else, like A – 10), we have *contextualized*—gone back to the context to see if we are representing the situation appropriately. And this does not apply only to symbolic representations. Suppose we drew a diagram to represent the situation, to help thinking:

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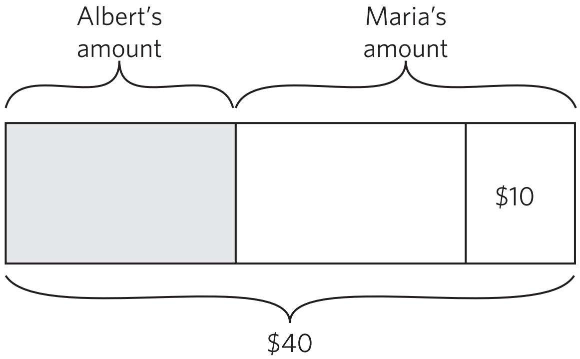
[7] J. Smith and P. W. Thompson, "Quantitative Reasoning and the Development of Algebraic Reasoning," in *Algebra in the Early Grades*, eds. J. Kaput, D. Carraher, and M. Blanton (New York: Erlbaum, 2007), 95–132.



This, too, is an occasion for decontextualizing and contextualizing as the solver determines if this visual representation is an accurate portrayal of the situation described.

More generally, visual representations present a powerful tool for supporting the use of these different forms of reasoning. Here are some examples.

Visual Representations in Quantitative Reasoning and Problem Solving

Just as geometric pictures can be used to help students—especially ELs—become more adept at noticing, reasoning about, and describing properties and relationships, so too can diagrams be useful for noticing, reasoning about, and describing quantities and relationships between them. Diagrams are a bridge between the words of tasks and their solutions, helping students by linking the relationships between quantities in the problem with the mathematical operations needed to solve the problem.

Further, diagrams can provide teachers a wonderful vehicle for prompting students to notice relationships between quantities. Read the Sharing Candies problem below, and then work to solve it. If your inclination is to solve it without diagrams, do so, and then go back and try it with diagramming:

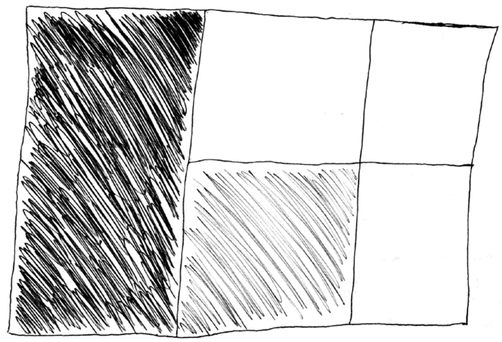
Sharing Candies

Sara had a bag of candies. She gave $\frac{1}{3}$ of the candies to Raul. Then Sara gave $\frac{1}{4}$ of the candies she had left to Jasmine.

After giving candies to Raul and Jasmine, Sara had 24 candies left in her bag. How many candies did Sara have at the beginning?

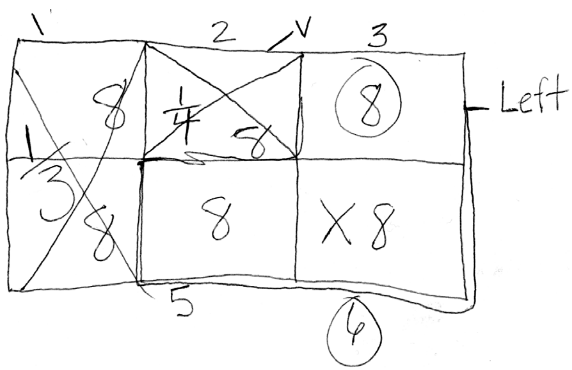
Create a diagram that helps you to solve the problem. Show your work.

In one classroom, the teacher had student groups (including ELs) present their solutions. Here is an example of the diagrams that students created and shared:



This example is color coded—in this case the rectangle on the left in black and the square in the bottom middle in gray—so not only are the amounts of candy given to Raul and Jasmine displayed clearly but so is the quantitative relationship between the two (Jasmine's amount was half of Raul's). Furthermore, an EL student presenting about a diagram like this one said, "We could see from this that Sara gave away the same number that she kept," to which one student from another group exclaimed, "I didn't see that!" Since Sara kept twenty-four, that meant she had started with forty-eight candies. This demonstrated that diagrams occasionally can reveal relationships that may not be apparent in totally symbolic approaches to the tasks.

Using diagrams is valuable in that the diagrams can, with minimal use of words, tell a story about how thinking has progressed during problem solving, and these stories can reveal EL student thinking to teachers in ways that words may not. Here is an example of a type of diagram we saw in EL student work on Sharing Candies:



A narrative unfolds from a diagram such as this one that may go beyond the few phrases an EL might write. The story told by this diagram can be verbalized as: *First, divide a rectangle into 3 parts and denote one of the parts as "$^1/_3$"; then divide the remaining part of the rectangle into 4 parts and denote one part as "$^1/_4$." The other 3 parts are what is "left," so draw an outline around them. Go back and divide the "$^1/_3$" part into 2 equal parts. Now the original rectangle is divided into 6 equal parts. 3 of those parts total 24, which is what Sara had left, so each small part represents 8 pieces of candy. With 8 pieces in each of the 6 parts of the original rectangle, that says that the total amount Sara started with is* ***48 pieces of candy****.*

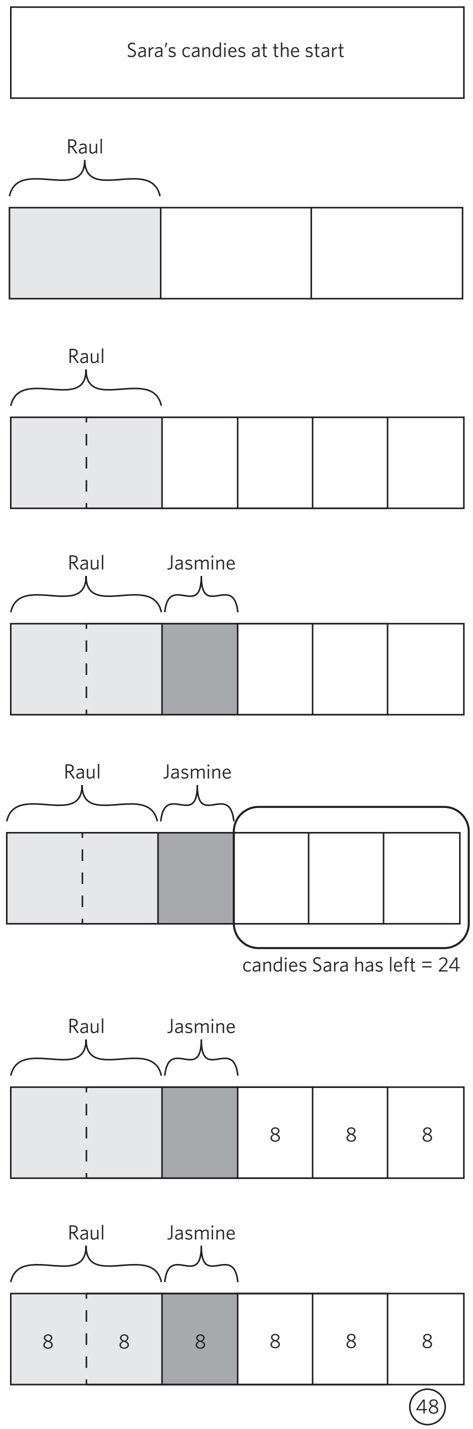
Visual Representations in Analytic Reasoning

Because the use of mathematical visual representations *as thinking tools* is foreign to most of our middle graders, we faced a steep uphill struggle to make them part of the fabric of mathematics learning for ELs. We began to see that analytic reasoning was a critical piece in solving this puzzle. By helping students become more adept at analyzing visual representations, we reasoned, teachers would be fostering students' own use of visual representations as thinking and problem-solving tools. This can occur when teachers ask students to present their methods of solution to the rest of the class, with visual records of their thinking, such as diagrams, on the board, screen, or chart paper. Teachers can stimulate analysis by asking pointed questions, such as "Where in that diagram do you see the $^1/_4$ of what Sara had left?"

In addition to these exercises in analyzing each other's thinking and solving, students can analyze fictional students' use of visual representations as tools for solving problems. This is a variation on so-called *worked examples*, a research-based strategy that has proven fruitful in teaching problem-solving strategies.[8] For example, students might be shown the Sharing Candies problem (assuming they'd not seen it before), then shown the work of a student named Janet, one step at a time. In pairs, as the steps are revealed, they write answers to prompts such as: "What changed from step 1 to step 2?" "What changed from step 2 to step 3?" and so on. After the last step, each pair writes its answer to the question: "What did Janet discover?"

[8] See, for example, J. L. Booth, K. E. Lange, K. R. Koedinger, and K. J. Newton, "Example Problems That Improve Student Learning in Algebra: Differentiating Between Correct and Incorrect Examples," *Learning and Instruction* 25 (2013): 24–34.

Janet's Work on Sharing Candies



The use of worked examples in this way, combined with opportunities for students to provide mathematical explanations for those worked examples, give them a chance to engage in analytic reasoning, thinking about how those problem-solving steps help to solve the mathematics task. Of course, we do not mean that a problem-solving procedure is provided and then students copy it! Laboratory research suggests that students use novice strategies (e.g., trial and error) when presented with traditional practice exercises but employ more efficient problem-solving strategies and rely on structural aspects of problems when presented with worked examples before solving (Cooper and Sweller 1987).[9] Worked examples support problem-solving transfer by developing students' understanding of how to reach a solution and methods of problem solving (e.g., Booth et al. 2013,[10] Cooper and Sweller 1987,[9] and Sweller and Cooper 1985[11]).

We chose a variation on worked examples because they can reduce cognitive overload; students can devote less working memory to the detail of how to get to a solution and instead focus on planning their work, describing their analysis of the visual representations and making connections to other tasks (Cooper and Sweller 1987).[9] Approaches like Janet's work provide scaffolds to structure students' engagement in mathematics and language and limit the verbal- and pictorial-processing demands on working memory. We do not advocate for using only worked examples—students must have varied and frequent opportunities to create visual representations themselves for solving tasks—but we believe that using them occasionally promotes analytic reasoning by students and can be a useful tool in developing students' understanding of how to use visual representations in problem-solving contexts.

Visual Representations in Spatial/Geometric Reasoning

The use of visual representations is perhaps most obviously connected to spatial/geometric reasoning, because geometric pictures often accompany

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[9] G. Cooper and J. Sweller, "Effects of Schema Acquisition and Rule Automation on Mathematical Problem-Solving Transfer," *Journal of Educational Psychology* 79, no. 4 (1987): 347–362.

[10] J. L. Booth, K. E. Lange, K. R. Koedinger, and K. J. Newton, "Example Problems That Improve Student Learning in Algebra: Differentiating Between Correct and Incorrect Examples," *Learning and Instruction* 25 (2013): 24–34.

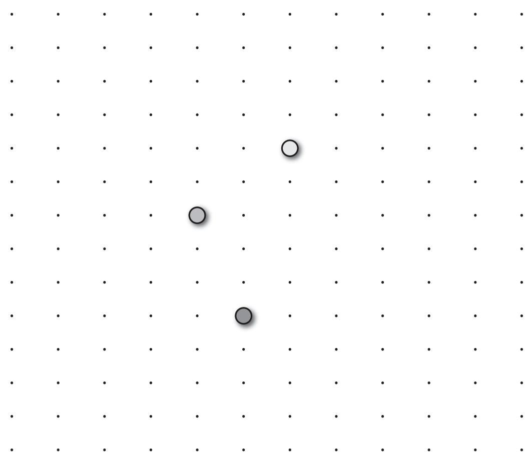
[11] J. Sweller and G. Cooper, "The Use of Worked Examples as a Substitute for Problem Solving in Learning Algebra," *Cognition and Instruction* 2, no. 1 (1985): 59–89.

tasks. In order to foster spatial/geometric reasoning among middle graders, we have used two different strategies.

Strategy 1. We have engaged students, including ELs, in tasks that invite reasoning about geometric properties, in addition to those that demand reasoning about geometric measurement, such as area and perimeter. An example is:

Parallelogram Problem

A parallelogram has three of its vertices at the three points shown on this grid. Draw a fourth point on the grid that makes the four points the vertices of a parallelogram. Once you have found one point that works, find another. How many can you find?



Strategy 2. For tasks that do involve area, perimeter, or volume measurement, where spatial/geometric reasoning is not typically used by students, we again use a worked example, by showing how fictional students have reasoned out answers to geometric measurement tasks by making use of relationships between various geometric figures. (For an example, see the Visual Representations Routine activity in Seminar 6 of the professional development materials associated with this book.) In essence, then, we employ analytic reasoning tasks to open the door to spatial/geometric reasoning.

The Standards for Mathematical Practice

We separate out these different categories of mathematical reasoning to underscore the connections with the Common Core Standards of Mathematical

Practice (SMP). The SMP describe the mathematical thinking, or habits of mind, in which proficient doers of mathematics engage. The SMP are essential for all students, and we must therefore work to provide access to the SMP for ELs. By working on supports for problem solving and analytic reasoning, in contexts that promote spatial/geometric reasoning and quantitative reasoning, we are able to engage ELs and other students in the SMP. Problem solving is addressed directly in SMP 1 (*Make sense of problems and persevere in solving them*), while analytic reasoning is essential to SMP 3 (*Construct viable arguments and critique the reasoning of others*). When working with visual representations in contexts that support quantitative reasoning, SMP 2 (*Reason abstractly and quantitatively*) is a natural fit because mathematical diagrams aid in moving back and forth between the quantities, their relationships, and what they represent. Spatial/geometric reasoning contexts can propel SMP 7 (*Look for and make use of structure*) because the geometric structure (e.g., the relationships between different geometric figures or components of those figures) become important. Furthermore, we emphasize explicit language strategies and the use of visual representations in part because they elicit, respectively, SMP 6 (*Attend to precision*) and SMP 5 (*Use appropriate tools strategically*).

We will discuss more about the Standards for Mathematical Practice and how they relate to the work of ELs in Chapters 3 and 4.

A Design Framework for Creating Access

The following four ingredients constitute our design framework for creating access for ELs:

- Challenging mathematical tasks
- Multimodal representation
- Development of mathematical communication
- Repeated structured practice

A critical feature underlying this design framework is the belief that, in order for mathematics teachers to help the ELs in their classes, they should regularly integrate academic language development with visual representations to open access to challenging mathematical tasks. Instructional routines—an enactment of repeated structured practices—structure and power this integration. These four key ingredients must be used in concert to be most successful. In this book we will focus in on each ingredient in at least one chapter; below, we give a short introduction to each.

Challenging Tasks

Teachers regularly include challenging mathematics tasks in instruction of their students of all English language proficiency levels. We gauge the degree of challenge in mathematics tasks by how much they engage students in various combinations of problem solving, analytic reasoning, spatial/geometric reasoning, and quantitative reasoning.

Generally and informally, "productive struggle" in mathematics means doing genuine mathematical work, such as engaging in problem solving and other kinds of mathematical reasoning. Tasks induce productive struggle to the extent that the questions "Is real mathematical work being done?" and "Who is doing it?" can be answered as "The *students* are doing the mathematical work on the task, and it is work in which they need to reason, or conjecture, or make a viable argument, and so on."

We will address this topic in detail in Chapter 5. For now, it suffices to say that this principle is based on a strong foundation, namely, the findings in the QUASAR study of urban middle-school mathematics classrooms, which showed that a regular diet of mathematics tasks with high cognitive demand improves student performance across the student population, including EL students.[12]

Multimodal Representations

Teachers use and promote multiple modes for expressing mathematical reasoning with their students of all English proficiency levels. In particular, they use, and help ELs use, mathematical visual representations. Classroom environments making ample use of multiple modes—pictures, diagrams, presentations, written explanations, and gestures—afford ELs the means first to understand the mathematics they are engaged with, and second to express the thinking behind their reasoning and problem solving. We will discuss multimodal representations, particularly visual representations, further in Chapter 2, but for now we want to emphasize that engaging students through a variety of modes, especially the nonverbal, like gestures and

[12] M. Henningsen and M. K. Stein, "Mathematical Tasks and Student Cognition: Classroom-Based Factors That Support and Inhibit High-Level Mathematical Thinking and Reasoning," *Journal for Research in Mathematics Education* 28, no. 5 (1997): 524–49.

mathematical visual representations, can provide non-English speakers the access and engagement that they need to succeed in mathematics.

Development of Mathematical Communication

Mathematics teachers help their students of all English proficiency levels develop the ability to communicate (by reading, writing, speaking, and listening) about mathematics. This includes both informal language used to explain mathematical thinking and the development of academic language, which are developed by *using* both informal and academic language to talk about mathematics. Teachers can do so by employing language access and language-production strategies that are integrated with the mathematical goals of lessons.

For one example: In the original version of the Sharing Candies problem, we feared the context (children sharing marbles on the playground), and possibly the wording, might be unfamiliar to students from other cultures, and we worked to make the context and language accessible to ELs. Even after those revisions (which resulted in the task you see), we were concerned about accessibility, so we wove a language access strategy into the implementation of the task, called the "Three Reads" strategy:

Sharing Candies

Sara had a bag of candies. She gave $\frac{1}{3}$ of the candies to Raul. Then Sara gave $\frac{1}{4}$ of the candies she had left to Jasmine.

After giving candies to Raul and Jasmine, Sara had 24 candies left in her bag. How many candies did Sara have at the beginning?

Create a diagram that helps you to solve the problem. Show your work.

1st Read	**CONTEXT** The problem is about _______________.
2nd Read	**PURPOSE** I need to ___________________________.
3rd Read	**INFORMATION** • •

In employing this strategy, the teacher leads the students through three readings of the problem, each time having them write down their answers to a different question: "What is the problem about?" (e.g., candy, or sharing candy, or sharing candies with two friends, and so on); "What is the problem asking you to find?" (how many candies Sara had in the beginning); and "What is some important information given?" (E.g., she gave $1/3$ to Raul; she gave $1/4$ of what's left to Jasmine; 24 left in the bag). This strategy came to mind because EL specialists advise teachers to "slow things down" when access by ELs is at risk—as with the risk of cognitive overload presented by a word problem. The Three Reads strategy operationalizes the advice to slow things down. Teachers and students—including native English speakers—appear to like it. In fact, one student in the class of one of our collaborating teachers asked his teacher if he "could use Three Reads on the MCAS" (the Massachusetts state examination). (See Chapter 4 for more information about the Three Reads and other language access instructional strategies.)

Repeated Structured Practices

Teachers engage students of all English proficiency levels in instructional routines designed to help the teacher focus on the use of challenging mathematical tasks, multimodal representation, and development of academic language.

Routinizing—that is to say, repeatedly using the same instructional sequence—allows teachers to focus more on their students and the mathematical content with which their students are grappling. Both teachers and students become familiar with the steps of the routine and do not need to spend as much cognitive effort on the mechanics of enacting the routine. Teachers can instead focus on how particular strategies support particular students (e.g., students who are ELs) and what their students are learning. They can take opportunities to "tinker" with the strategies embedded in the routine to learn what works best for students. In this manner, using mathematical instructional routines contributes to building teacher capacity for meeting students' needs.

Our use of the term "instructional routine" is drawn from a growing literature on "well-designed procedures that have been proven in practice, that take account of the complexity of the goals that need to be accomplished, and that allow the practitioner temporarily to hold some things constant while working on others (130)."[13] Instructional procedures are intended to help teachers manage complexity in instruction, such as adapting to what they observe in students' work during lessons. In mathematics, they are generally designed to provide opportunities for students to reason about mathematical ideas.

We will discuss routines further in Chapter 6.

Summary

In our work with mathematics teachers with students who are ELs, all of our efforts are based on a commitment to *access* for ELs to opportunities for productive engagement with mathematics.

You may have heard a sentiment we have heard, namely, that in meeting the challenges in ensuring ELs' success in mathematics, "mathematics teachers must be language teachers as well." There is truth to the statement, but that particular framing makes it sound as if teachers of mathematics must sign up for an additional career. Rather, we prefer to argue that mathematics teachers need not veer from their chosen profession. We deeply believe that our four principles, which undergird this entire book, can make it possible to foster mathematics success for ELs from within the *learning and doing of mathematics*.

Even with our arguments and examples, the four design principles may still seem abstract, so we close this chapter with quotes from two teachers who worked with us, as they looked back at what they had learned and what they noted about ELs in their classes:

> [S]eeing the different ways that they [the students] can use diagrams, just even having that idea of a diagram in their head and maybe having a couple of different pictures of what could be a diagram; [. . . it] can help them to then sometimes use it as a strategy anytime they are getting stuck.

[13] M. Lampert, H. Beasley, H. Ghousseini, E. Kazemi, and M. Franke, "Using Designed Instructional Activities to Enable Novices to Manage Ambitious Mathematics Teaching," in *Instructional Explanations in the Disciplines*, eds. M. K. Stein and L. Kucan (Springer Science+Business Media, LLC, 2010).

> As time progressed, they understood the value of it (diagramming). They recognized that it was a math tool and not an art project. And they got really excited when they explained someone else's diagram. And that is where it registered for me that they understood what they were doing.

In the next chapter, we explore more deeply the use of mathematical visual representations to create greater access for English learners to mathematical thinking and mathematical communication.