Patsy Kanter + Steven Leinwand

developing numerical fluency

Making Numbers, Facts, and Computation Meaningful

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Dedicated to Ernie and to Ann.
They keep us centered,
honest, and very happy.

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Introduction

We have been teaching mathematics to children and to their teachers since the early 1980s. We encourage teachers to see the power of student discourse to foster engagement, models to support visualization, contexts to connect the mathematics to the real world of children, and alternative strategies to effectively differentiate. We have seen how these strategies, all of which have the support of a growing body of research and the wisdom of good practice, have changed the trajectory of student learning of mathematics. These same strategies apply directly to the challenge of ensuring the development of numerical fluency in all elementary school students.

Defining Numerical Fluency

Join us on the rug with twenty-two second graders. "Ready?" we ask. "Turn and tell your partner how much eight plus nine is." After about ten seconds we add, "Call out what you said to, or heard from, your partner when we point to you." We hear a chorus of "seventeen" with a "sixteen" or an "eighteen" here and there, and even a stray "one" mixed in. "Great. Again, with your partner, use words, numbers, or pictures on your whiteboards to convince the class that eight plus nine really is seventeen." Here is what we often hear:

> *I knew that eight plus eight is sixteen, and because nine is one more than eight, eight plus nine has to be seventeen.*
>
> *We decomposed the eight into seven and on and added the one to nine to get ten and then seven more for seventeen in all.*
>
> *I drew eight tallies on top and nine tallies on the bottom and counted them all to get seventeen.*
>
> *I did a similar but different way by adding one to the nine and taking one away from the eight so that eight plus nine was the same as seven plus ten or seventeen.*
>
> *We used a double too, but we thought that since nine plus nine was eighteen, eight plus nine had to be one less or seventeen.*
>
> *I thought of ten-frames and since eight is a ten with two holes and nine is a ten with one hole, eight plus nine had to be ten plus ten or twenty minus the three holes, or seventeen.*

This is when and why teaching mathematics is such a joy! This is a small example of what we consider to be teaching that develops fluency and students who can demonstrate that fluency. Even incorrect answers or explanations are just as valuable as these correct answers, as we hear students argue about who is right, why they disagree, and then often correct themselves.

What Fluency Is

In the K–5 *Common Core State Standards for Mathematics* (National Governors Association 2010), the word "fluently" appears only *seven* times:

- Grade K: Fluently add and subtract within 5.
- Grade 2: Fluently add and subtract within 20 using mental strategies.

- Grade 2: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
- Grade 3: Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division or properties of operations.
- Grade 3: Fluently add and subtract within 1000, using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
- Grade 4: Fluently add and subtract multi-digit whole numbers, using the standard algorithm.
- Grade 5: Fluently multiply and divide whole numbers, using the standard algorithm.

We disagree with one point in these standards: the notion of *the* "standard algorithm" is one of very few glaring errors in the Common Core and needs to be read and implemented as *a* "standard algorithm" that encourages the teaching of alternatives that are much more accessible to many students. We will discuss this more in Chapters 6 and 9.

Note the relationship between fluency and *mental strategies,* and between fluency and *place value* and *properties*. Fluency is tied directly to *conceptual understanding* of numbers, place value, and operations, not simply to rote memorization of procedures. As the National Council of Teachers of Mathematics said in *Principles to Actions*:

> Fluency is not a simple idea. Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meaning and properties of the operations. (NCTM 2014, 42)

Based on all this research, we argue that:

- Numerical fluency is about understanding, not memorization.
- Numerical fluency is supported by fingers, pictures, and all sorts of materials—that is, multiple representations that best fit individual students' conceptions.
- Numerical fluency develops when students communicate their understandings and construct and share their strategies.
- Numerical fluency is developed much more through engaging tasks and activities with rich questions and student discourse than through endless practice with mindless "get-the-answer" worksheets.

What Fluency Is Not

Unfortunately, this broader conception of fluency is very different from what we find in many classrooms and helps to explain why so many students struggle with mathematics in elementary school. We often hear a range of definitions that neither we, nor the research, support:

- Fluency is instantaneous recall.
- Fluent students do not have to use their fingers.
- Fluency is a good indicator of a student's mathematical ability.
- Fluency is correctly identifying the answers to 100 facts in a very short time.
- Fluency is the ability to quickly and accurately add, subtract, multiply, and divide with pencil and paper.

These are the deeply held beliefs and mindsets that guide instruction in too many classrooms. It is our conviction that these beliefs are counterproductive and even very destructive for many of our students.

The Challenge: Teaching in Ways We Were Not Taught

As we travel around the country, we are excited to see that fluency has been elevated in school-based discussions and has drawn the attention of many teachers. Done correctly, the development of numerical fluency empowers students and helps them develop lifelong understanding and confidence. Done wrongly or poorly, we deprive students of mathematical empowerment and send them messages that severely undermine their long-term mathematical development. We see exceptional teachers who are fully committed to teaching conceptually, but who still resort to speed tests because they see no alternatives. We see good mathematics programs that fail to give teachers structures for instruction or effective strategies that develop fluent mathematicians. And the questions we get remain essentially the same:

- "How do I teach number facts so that students know their facts with fluency?"
- "How do I help students who only want to follow rules that they obviously don't understand to compute accurately, efficiently and with confidence?"
- "Why do my students have such difficulty estimating sums, differences, products, and quotients?"

We believe that the core of the problem is that most teachers today learned mathematics as routines and procedures. When asked for the sum of eight and five, they have no problem answering "thirteen," and explaining that this is simply a "fact I have memorized." But when asked to apply this knowledge to 38 + 5—clearly 30 more than 13, or 43, calculated with elementary place value understandings—these same teachers ask for pencil and paper and resort to procedures including "and carry the one." These teachers (and many of our students) do not have numerical fluency. They know their facts, but have not acquired the critical ability or disposition to think and reason numerically in ways that represent the fluency called for by the Common Core. Simply put, the facts are essential, but not enough.

In addition, many teachers are the products of instruction that made regular use of timed tests of facts. They struggle to find other, more effective, ways to build students' fluency. Unfortunately, this keeps the focus on rote memory and speed, not on understanding and the powerful thinking and visualization strategies that undergird true fluency. As a result, far too many students, early in their mathematical careers, learn that mathematics is arithmetic and that arithmetic is the mindless memorization and regurgitation of facts and procedures that don't have to make sense and that don't relate to the bigger aspects of conceptual understanding.

Instead, we envision teaching that starts with that conceptual understanding first, and builds fluency from it. Figure I.1 summarizes what this kind of teaching looks like—what teachers and students are doing when this kind of work is going on.

Build procedural fluency from conceptual understanding (NCTM 2014, 47–48)

What are ***teachers*** doing?	What are ***students*** doing?
• Providing students with opportunities to use their own reasoning strategies and methods for solving problems • Asking students to discuss and explain why the procedures that they are using work to solve particular problems • Connecting student-generated strategies and methods to more efficient procedures as appropriate • Using visual models to support students' understanding of general methods • Providing students with opportunities for distributed practice of procedures	• Making sure that they understand and can explain the mathematical basis for the procedures that they are using • Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems • Determining whether specific approaches generalize to a broad class of problems • Striving to use procedures appropriately and efficiently

Figure I.1 *Teacher and student actions to build procedural fluency*

There is an unacceptable chasm between traditional mathematics instruction, that rarely works for more than one-third of our students, and this kind of mathematics instruction, that truly empowers nearly all students.

Our Goals for This Book

Our goals in the chapters that follow are to provide a broad range of ideas, techniques, activities, and strategies for converting these teacher and student actions into ongoing classroom practices. The questions we pose, and answer, in this book include:

- How can fluency be developed more effectively?
- What specific strategies and techniques can teachers use to develop numerical fluency in all students?
- Why are these approaches so important for building a stronger foundation for all mathematical learning?

To answer these questions we begin by defining numerical fluency in terms of what we believe it is, and, as importantly, what it is *not*, trying to sweep aside a range of misconceptions about fluency and how best to develop it. Upon the definitional foundation laid out above, we start in Chapter 1 with a set of nine pivotal understandings that undergird all of our instructional work to develop numerical fluency. We believe that these understandings form a powerful hierarchy of skills and concepts that help teachers diagnose diverse readiness; differentiate as needed; and place instruction, reteaching, and reinforcement along a common continuum from understanding parts and wholes to a mature sense of place value and operations.

But as we have learned from both the *Principles and Standards for School Mathematics* (NCTM 2000) and the *Common Core State Standards for Mathematics* (National Governors Association 2010), student processes or practices are as important as, and intertwined with, the mathematical content itself. Accordingly, we turn in Chapter 2 to six critical processes for the development of numerical fluency that characterize all effective instruction: contextualizing, constructing, representing, visualizing, verbalizing, and justifying. These six processes, in combination with the ten pivotal understandings, create the weft and warp of our numerical fluency tapestry. Then in Chapter 3 we propose and describe a set of classroom structures that support classroom instruction demonstrating this tapestry of numerical fluency.

Chapters 4, 5, and 6 apply this framework to the development of addition and subtraction fluency, and Chapters 7, 8, and 9 apply it to the development of multiplication and division fluency. Each chapter identifies and describes a set of big ideas that unify good instruction, and provides a range of examples and vignettes drawn from our classroom experiences. Finally, in Chapters 10 and 11 we offer suggestions for some easy-to-implement schoolwide activities and resources and the professional development and collaboration that are required to ensure that it is the entire elementary school, and not just individual teachers, that shares the responsibility for developing numerical confidence, joy and fluency in every student.

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Chapter 2

Six Processes for Developing Numerical Fluency

As we noted in Chapter 1, students do not develop numerical fluency by memorization and regurgitation of rules. Rather, numerical fluency develops over time as students engage in active thinking and doing. They must strategize, reason, justify, and record and report on their thinking. Accordingly, the effective development of numerical fluency involves the use of a set of cognitive processes throughout mathematics, not just in one lesson or introductory lessons. When mathematics lessons are systematically planned and implemented with these six processes at the forefront, teachers maximize the chances of *all* students becoming numerically fluent and mathematically powerful!

The Six Processes

We have identified six processes that support the development of numerical fluency. Figure 2.1 summarizes the six processes in student-friendly terms and in more formal terms.

> These processes are not unique to numerical fluency—in fact, the same processes are essential for the development of spatial sense, algebraic reasoning, and other big ideas in mathematics.

Formal cognitive process	Student-friendly terminology
Contextualizing	Storytelling
Physically constructing	Building
Representing graphically and symbolically	Drawing and using symbols
Visualizing	Seeing
Verbalizing	Describing what and how
Justifying	Discussing why

Figure 2.1 *Six processes for developing fluency*

1. **Contextualizing or Storytelling:** Understanding that life can be described mathematically is at the foundation of fluency. Equations exist because they are a shortcut to explain situations, look at reality, and make predictions. Too often we present equations without giving them context, leaving children without understanding and causing misconceptions.
2. **Physically Constructing or Building:** Children need to manipulate materials to develop an understanding of the action of operations, which is then extended to the visual or pictorial level and then to abstraction. Fluency demands multiple models and making connections between and among models.
3. **Representing Graphically and Symbolically or Drawing and Using Symbols:** Seeing and using models and relationships between models supports visual memory, building relationships, and mental fluency, and enhances long term memory. For many students, fluency depends on being able to visualize concepts in different ways and understand the relationships between these different representations.
4. **Visualizing or Seeing:** Children learn to visualize quantities and the relationships between them. As children accumulate a visual repertoire, their numerical fluency grows because they are able to "see" the mathematics in which they are engaged.
5. **Verbalizing or Describing What and How:** Understanding of operations is achieved when students describe and explain what they did with the materials that they manipulate and the pictures they draw. Students need

to describe the representations they create and how various representations are similar and different. Students should always be expected to describe the "what" and "how" of the mathematics they are learning, using informal and, gradually, formal mathematical language.

6. **Justifying or Discussing Why:** Discussing relationships and justifying solutions to problems is fundamental to developing metacognition and crucial to long-term fluency. Justifying answers the question "why?" and is one of the best ways to monitor the development of numerical fluency in students.

These six processes emerge from an expansion and elaboration on the core "concrete to representational to abstract" trajectory well-known to elementary teachers. All we've done is adapt "concrete" to become "construct" or "build" with physical materials, and focus "representational" on actual drawing and pictures. We also expand "representational" to include visualizing or abstracting the representations. And we add contextualizing in the form of storytelling, and replace "abstract" with verbalizing and justifying.

Fluency emerges from diverse experiences that link numbers and operations to contexts and familiar situations, and that provide students throughout their mathematical development with opportunities to construct, visualize, verbalize, and justify. Let's look at each of these processes in turn and see how each plays a critical role in supporting the development of numerical fluency.

> These six processes are the essence of the differentiation needed to effectively teach mathematics. One student might make sense of subtracting in a context, while another needs to touch and feel the comparison of two qualities, while a third learns best by "talking through."

Contextualizing or Storytelling

Storytelling and contextualizing serve many purposes:

- Students develop a firmer understanding of the meaning of the operations by situating them in context. For instance, thinking about Sarah who had 4 apples and bought 3 more helps students understand the "adding on" meaning of addition.

- Through storytelling, students have the chance to act out and visualize situations, providing the opportunity for two other important processes: constructing and visualizing (see more on these below). The context provides a starting point for students to construct and visualize the changes and relationships in the problem.
- Asking students to create stories from numerical expressions helps students to make sense of numbers and operations and helps them make critical connections between abstract mathematics (3 × 7) and the real world (the area of a three cm. by seven cm. rectangle or the cost of three t-shirts that each costs $7).
- And let's be honest. How many students really care about the product of 8 and 32, when the alternative is a YouTube video of the pandas in a zoo eating, accompanied by data like "eight pandas who each eat about thirty-two pounds of bamboo each day"? Such a context and the story about how many pounds of bamboo are needed to feed the pandas support engagement, smiles, and learning far more effectively than a workbook page of twenty naked multiplication problems where the goal is simply arriving at correct answers regardless of the depth of understanding.

What Contextualizing Looks Like

A kindergarten teacher writes on a flip chart (see Figure 2.2) and asks students to turn and tell their partners what they see. This question uses representations and verbalizations to stimulate various descriptions of the number 5 and is augmented by a discussion of how the three representations are the same and how they are different. But the critical link between the number 5 and the many contexts in which it appears emerges from the follow-up task posed to the students: "Stroll around our classroom with a partner and see how many places you can find this number, and how you know."

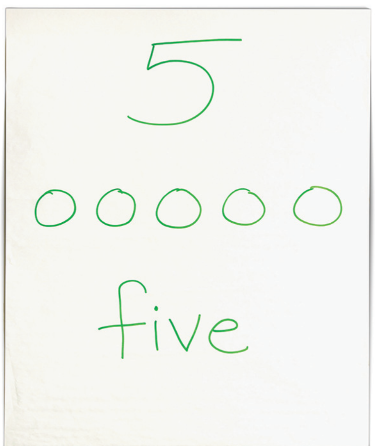

Figure 2.2 *A chart for 5*

Students are contextualizing and connecting the abstraction of "five" as a numeral, a set of circles, and a word to their personal environments as they find and describe 5 on the clock, on the calendar in 5, 15, and 25, on the number line above the white board, with five books, five cubes, five fingers, or five markers. Needless to say, this ability is a critical precursor to operating with numbers and solving word problems.

Contextualizing also occurs when naked numbers (plain numbers with no context) are replaced with data or information, as in shifting 4 + 3 to four apples and three apples. We are great fans of using money, menus, price lists, and other familiar contexts as platforms for doing mathematics. And students contextualize every time we ask them to create their own story problems (as in Figure 2.3).

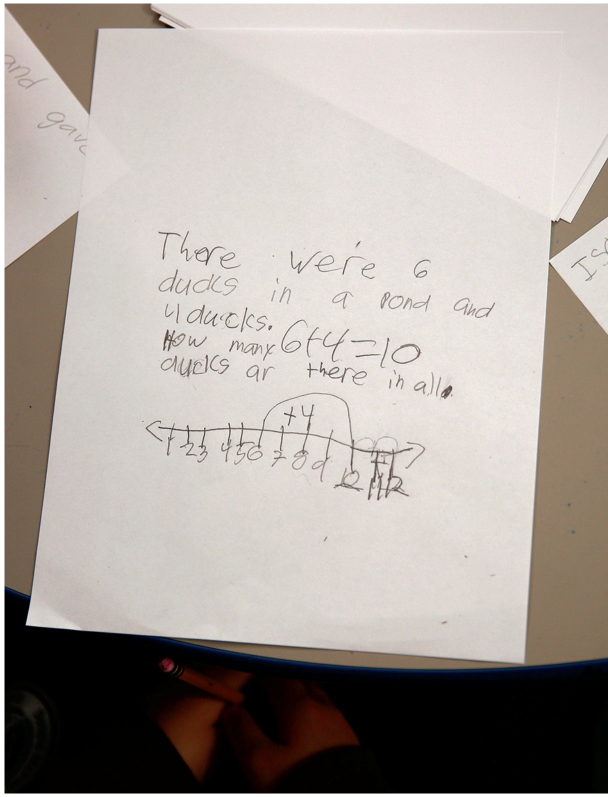

Figure 2.3 *Creating story problems with number lines*

Constructing or Building

Building with objects allows children a unique opportunity to create the structure of numbers and operations for themselves. No worksheet can teach a child place value as effectively as giving that child 153 beans, a place value mat, and fifteen cups to make tens. It is the process of grouping those ten beans into one cup that helps children understand how ten ones become one ten. This concept is fundamental, hard to grasp, rarely recognized as essential, and yet lies at the very core of understanding number relationships.

Similarly, young children's constructions help them create innate understandings. For example, when young children use number tiles to make 6 in different ways, with the rule that tiles must touch "full side to full side", they have the opportunity to explore many different ways to physically represent the number 6 (see Figure 2.4). Then, when students are asked to find how many different ways they can arrange the tiles, they need to determine that each configuration is unique. Out of these constructions, owned by the students, come insights that 2 and 4 make 6, 5 and 1 make 6, 3 and 3 make six, and so on, setting the stage for an understanding of part-part-whole relationships. This happens because the child built the arrangements and talked about what they built.

Constructing with concrete materials and objects is often a necessary prerequisite for drawing diagrams and pictures where it is more difficult to manipulate and rearrange the objects.

Figure 2.4 *Ways of arranging six square tiles*

What Constructing Looks Like

Tell a class of first graders about the child looking out of his or her bedroom window and seeing four birds in the branch of a tree. Next state with amazement that then the child looked down at the ground and saw three more birds! What seems so obvious—that there are seven birds in all—is very far from obvious to many children. Representing the four birds on the branch and the three birds on the ground with blocks or linking cubes is for many students a critical step in moving toward numerical fluency. Such activities are essential prerequisites for building an understanding of joining numbers, and can lead to drawings, visualizations, and fluently solving problems about joining and separating a group of four and a group of three. Note that here, as with nearly all of the examples we use, there is a natural reliance on more than one process, resulting in a combination of constructing, contextualizing, and verbalizing.

Representing Graphically and Symbolically or Drawing and Using Symbols

As students become comfortable using objects and materials to physically represent situations, we need them to be able to represent these mathematical situations using pictures, drawings, mathematical notation, and eventually equations, in addition to physical objects. Teachers often make the connection between constructing and drawing by asking students to replicate what they have already created with objects, but this time on paper with drawings. These representations constitute a pictorial and/or symbolic language that can and should help all students make sense of mathematics.

> Sharing different representations supports a deeper understanding of the power of multiple representations, because a representation that works for one student is unlikely to be the representation that works for all students.

When students move from drawings and pictures to representations like number sentences and equations, it is important to remember that equations are not merely tools to find answers, but are also symbolic mathematical representations that help to make mathematics so powerful. Drawings and other concrete and pictorial representations are not the end or the purpose, but rather key steps that connect to more abstract representations, like number sentences and equations (see Figure 2.5).

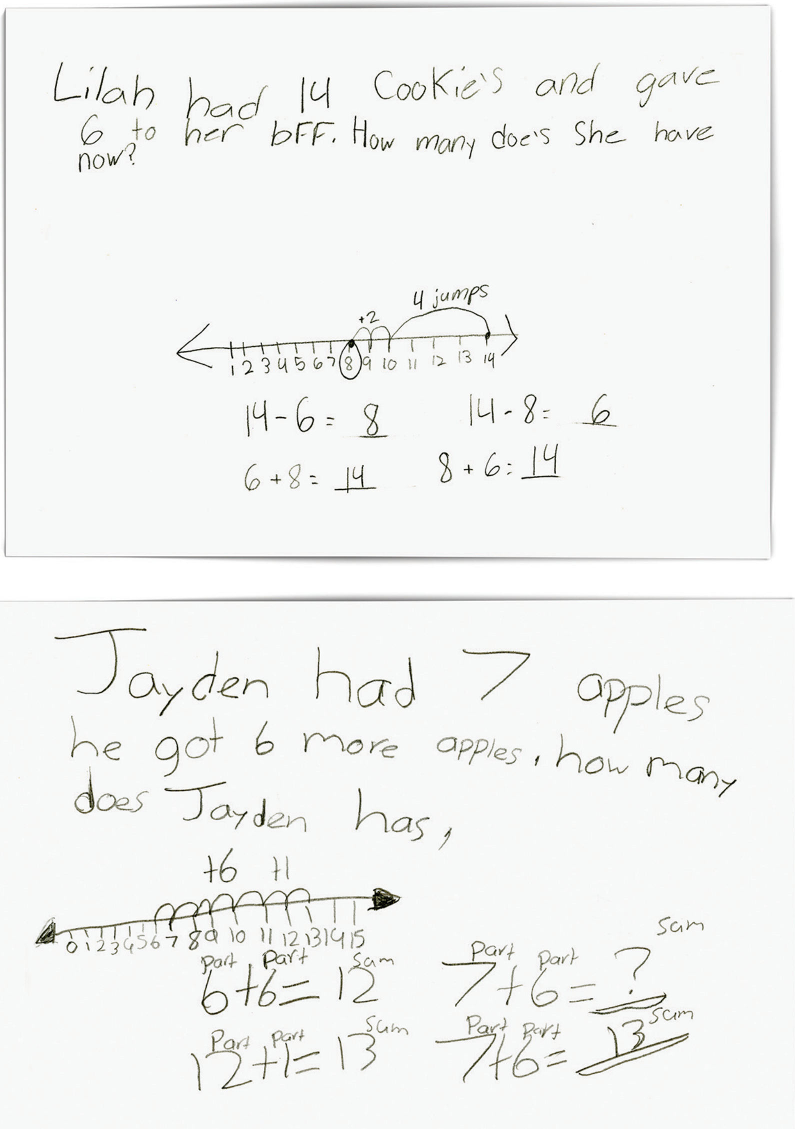

Figure 2.5 *Examples of combining abstract representations such as a number sentence and equations*

What Representing Looks Like

Representing includes understanding that signs for operations represent actions and that the equal sign means *equivalent* or *the same* and is *not* an operation or an action. First and second graders learn that the operation they see in put-together and add-on stories is represented with a + sign, and that what they do when comparing and taking apart or away is represented with a – sign. Likewise, students learn that finding the total of equal groups can be represented with multiplication symbols, and that splitting a whole into equal groups can be represented with division symbols.

Figure 2.6 shows some problems we've used with third graders that involve symbolic representations and help to build an understanding of operations and equality. Students who struggle with symbolic representations should be encouraged to use physical models to show the numbers before deciding whether to add or subtract.

Use +, –, = to make each of these a true number sentence.

1.	7	8	15	
2.	10	7	3	
3.	33	10	23	
4.	9	7	16	
5.	36	8	28	
6.	45	10	5	40

Figure 2.6 *Activity to build an understanding of operations and equality*

Visualizing

To paraphrase Jo Boaler (2016), mathematics is a subject that allows for precise thinking, but it is when that precise thinking is combined with creativity and visualization that the mathematics comes alive. When we don't ask students to think visually, we miss an incredible opportunity to increase students' understanding and to foster critical connections between the left and right hemispheres of the brain.

To visualize is to picture mentally—a process that draws on the right hemisphere of the brain to support the work of the left hemisphere. After many experiences with cubes, tiles, ten-frames and base ten blocks, we need to help students visualize what they might have drawn or constructed without actually drawing or constructing. In the world of fractions, this means that students are gradually able to build a "visualization repertoire" for "three-quarters" that might include three silver quarter coins, pizza, a measuring cup, window panes, or a ruler, each of which serves as a powerful visual representation with which to build fluency with fractions.

What Visualizing Looks Like

Here's what we observed in a third-grade classroom. "We've been using different representations to model our addition and subtraction problems. Now, here is what I would like you to do. For the problem 192 – 55, I want you to visualize how you can represent this problem and then describe what you see to your partner. For this task we won't be using materials or whiteboards or paper. We'll just be picturing mathematics in our heads. Then we will share what we see with the whole class."

Here is some of what the students reported that left us convinced that they were well on their way toward developing strong numerical fluency:

> *I saw a bar model. There was a bar with 192 on the top and a second bar with 55 underneath it. I know that I can add on to the 55 to find the difference.*
>
> *I pictured a number line with an arrow at 192. I drew an arrow to the left and wrote 55 on the arrow so that I knew I had to count back 55.*
>
> *I pictured base ten blocks. There was one hundred, nine tens, and two ones, and I pictured myself removing six tens and putting five back.*
>
> *I visualized money and saw a dollar, nine dimes, and two pennies. In my mind I changed one of the dimes into ten pennies and then I removed five dimes and five pennies.*

Verbalizing or Describing What and How

Verbalizing—or describing what you see or what you are thinking—is a crucial step in the transition from concrete and representational to abstract understanding. Verbalizing gives students the opportunity to hear themselves thinking and to share that thinking (see Figure 2.7). Building on constructions, graphical representations, and their visualizations, verbalizing adds language to the mix, thereby strengthening the connections within the brain among executive processing, visual, language, and memory functions.

Verbalizing serves four main purposes:

1. Verbalizing provides the opportunity to describe mathematical entities. For example, given the number sentence: $(7 \times 3) + 1 = 22$, students should be able to verbalize that "there were seven rows of three and I had one left over, so I had twenty-two."

2. Verbalizing allows children to confirm understanding of operations, first for themselves and then for others. For example, when asked "What is the meaning of $7 + 3 = 10$?" a first grader might verbally explain that the set of seven grew larger: "I had seven marbles and I got three more, so my answer is a bigger number. It is ten."
3. Verbalizing surfaces key vocabulary and asks students to use this vocabulary in their explanations.
4. Verbalizing also enables students and their teachers to focus on language and descriptions of how and what students are thinking. For example, after drawing a bar model to represent an addition problem, students might tell us that "Since the girl had fifteen cookies in all, I drew a rectangle and labeled it 'fifteen.' Then since she ate six cookies, I cut the bar into two pieces and made the smaller piece 'six.'"

Of course students use verbalization to explain their thinking and justify their answers, but we prefer to differentiate these more formal explanations or discussions of "why" from the descriptions of "what" and "how" that must precede justifying.

Figure 2.7 *Verbalizing adds language to the mathematical mix.*

What Verbalization Looks Like

Imagine a first-grade class asked to describe what they notice to their classmates about the arrangement of blocks shown in Figure 2.8. Notice the purpose is to "describe," not yet to "justify."

> Keisha says: *I see two rows of three and one more.*
>
> Cal says: *I can see three groups of two and one left over. I know there are seven.*
>
> Sam says: *I see a ten-frame showing seven.*

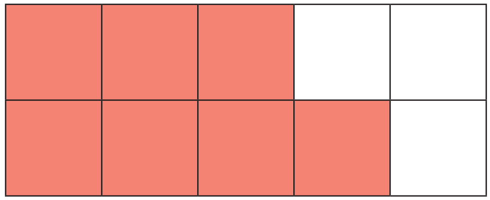

Figure 2.8 *Block arrangement*

> Amanda says: *I see some shaded blocks and three unshaded blocks.*

Note how verbalization and describing surface critical vocabulary, like "groups" and "rows" and "more," even in this simple example. Notice too how describing provided a natural lead-in to justifying and reminds us of the power of the simple "What do you notice?" question.

Justifying or Discussing Why

When we value student thinking as something that is just as important as simply getting correct answers, we need to create classroom norms that expect students to justify their work and their thinking. In fact, being able to justify one's answer is often more important than getting the right answer because the justification often reveals misunderstandings and surfaces mistakes. We have found that justifying is easily stimulated by such questions as:

> *Why do you say that?*
>
> *Why do you think that?*
>
> *Can you explain that, please?*
>
> *Who doesn't agree? Why not?*
>
> *Who did it differently and can explain how their approach is different?*
>
> *Convince us that you are right* (whether the answer is correct or incorrect).

When we ask students to justify their thinking or to convince the class that what they did or said is correct, we are attending to two critical Common Core Standards for Mathematical Practice: #2, "Reason abstractly and quantitatively";

and #3, "Construct viable arguments and critique the reasoning of others" (National Governors Association 2010). What capability is more important for our students in the future than an ability to construct and critique viable arguments in their writing, their mathematics work, their science learning, throughout social studies, and in life in general?

What Justifying Looks Like

Consider fourth graders talking about how they know that 23 × 7 is 161.

> *I know seven 25s is 175. So, 23 is two less than 25, so I multiplied 2 times 7 and got 14. I subtracted 14 from 175 and got 161.*
>
> *First, I know that the answer is going to have a 1 in the ones place because 7 times 3 is 21, so I think 161 is right.*
>
> *I multiply 20 times 7 and get 140 and 7 times 3 and get 21. 140 + 21 is 161.*

This justification can come verbally or in writing, and with visuals. The important point is that to be fluent, one must be able to justify one's answer and support one's justification with words, pictures, and/or symbols.

Putting the Six Processes Together

Consider presenting the table in Figure 2.9 to a class of third graders.

Vegetable or fruit	Number needed	Cost
Tomatoes	5	49¢ each
Cucumbers	3	85¢ each
Lettuce	2	$1.50 per bunch
Green peppers	4	69¢ each
Peaches	8	75¢ each
Apples	7	85¢ each

Figure 2.9 *Food order for salad and dessert*

Presenting and posting a table of data like this is the perfect stimulus for a "what do you notice?" and "what do you wonder?" discussion to build familiarity with a context and stimulate student interest. We hear students, after talking with their partners, announce to the class:

I see fruits and vegetables.

I see that lettuce is the most expensive.

I notice that you need less than ten of each item.

We see that you need the most number of peaches.

Rather than limit this problem to something as narrow as how much the apples cost, teachers who use "What do you wonder?"—a form of contextualizing—get to choose from such student-provided tasks as:

Which item do you spend the most money for?

Which item do you spend the least money for?

About how much does the entire order cost?

What does the entire order cost?

Within the context of this story or situation, and starting with "Which item costs the most?" teachers can lead students to strategize about which items cannot possibly cost the most and why—thereby supporting estimation skills. They can then ask students to focus on the apples and the peaches and ask students to use base ten blocks, a picture, and a number sentence to help them represent the situation—thereby focusing on multiple representations (including constructing, drawing, and visualizing). Teachers can then help students further develop numerical fluency through such questions as:

Can you explain how these representations are the same? (verbalizing)

How are they different? (verbalizing)

Turn and explain to your partner how you know how much the peaches will cost. (justifying)

The numerical fluency we are seeking is evident in a classroom where students share and explain such insights as:

If there were ten peaches, you would pay $7.50. But two fewer peaches would be $1.50 less—so the peaches cost $6.

I thought that 75 cents is seven dimes and five pennies. So I'd need 56 dimes and 40 pennies. That's $5.60 plus 40 cents or $6.

> *I knew that two peaches would cost $1.50, so four would be $3 and eight would have to be $6.*

Note how a table of data provides a context and how our questions provide students with the opportunity to construct, represent, visualize, verbalize, and justify. And this is only one of many problems and tasks our students can wrestle with, based solely on one data table and the notice-wonder launch (see Max Ray-Riek's *Powerful Problem Solving* [2013] for more on this) when the six processes guide our work.

Conclusion

We know what constitutes good instruction. It is evident in classrooms wherever students link mathematics to real situations, represent the situations with objects and/or drawings, describe what they see, and justify their thinking. These are the processes that need to be front and center wherever and whenever mathematics is being taught.