3 Relations and functions

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KEY CONCEPT

Form

RELATED CONCEPTS

Models, Representation, Systems

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

Decision-making can be improved by using models to represent relationships in different forms.

Factual

- How does mathematics describe the relations between objects?
- What representation can be used to describe relations?

Conceptual

- In what contexts are unique answers important?
- Are mathematical operations always reversible?

Debatable

- Does a visual representation of mathematics carry sufficient information and precision?
- Do mathematical relations have to follow an obvious rule?

Do you recall?

1 Find an equation of a line with y-intercept (0, 3) and gradient 2.

2 Find an equation of a line through the points (0, 4) and (4, 0).

3 Find the tenth term in the sequence 1, 4, 7, 10, 13, … using the obvious pattern.

4 Is 16 the only number that is 12 more than 4?

5 Is 4 the only number that can be squared to give 16?

6 If you are asked how many numbers there are between 1 and 10 inclusive, what needs to be said about the type of number to be sure of the answer?

3.1 Relations

3.1.1 Making a connection

Explore 3.1

Make a list of the countries you have visited.

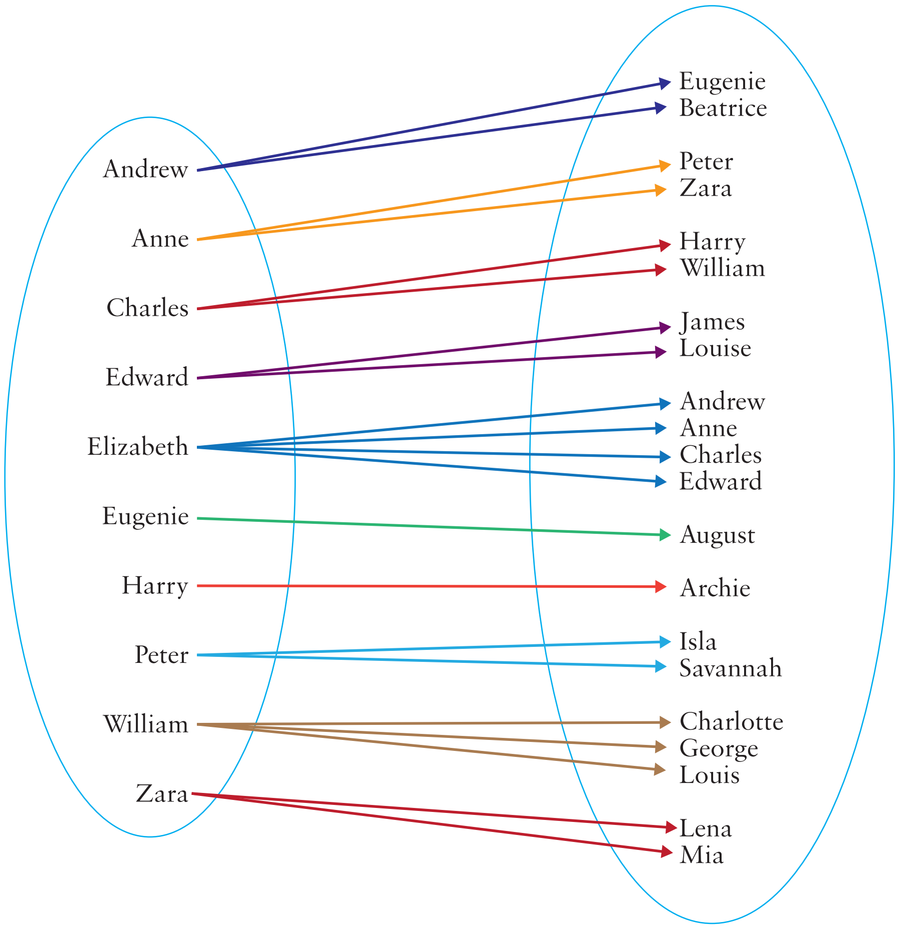
List all the members of your class on the left side of a page, all the countries any of you have visited on the right side, and then make connections between each person and their countries. Which country has the most links? Are there some countries with just one link?

How many members of your class can be uniquely identified from just the list of countries they have visited?

The mathematical idea of a **relation** means a link between two sets. It can be applied in many contexts and represented in a number of different ways.

is the parent of ↑ →	Andrew	Anne	Archie	August	Beatrice	Charles	Charlotte	Edward	Eugenie	George	Harry	Isla	James	Lena	Louis	Louise	Mia	Peter	Savannah	William	Zara
Andrew					×				×												
Anne																		×			×
Charles											×									×	
Edward													×			×					
Elizabeth	×	×				×		×													
Eugenie				×																	
Harry			×																		
Peter												×							×		
William							×			×					×						
Zara														×			×				

The grid on the previous page shows the parental relations between Queen Elizabeth II and her direct descendants in the British Royal family. For example, Zara is the parent of Lena and Mia.



This diagram is another representation of the same information. It is called a mapping diagram. The parents, on the left, form a set called the **domain**. The children, on the right, form another set called the **range**.

A relation links the elements of the domain and those of the range. Expressed in words, the relation can be stated as 'is the parent of'.

Because this concerns the direct descendants, one parent is shown in each family. However, one parent may have many children. This is an example of a **one-to-many** relation.

There are some elements of the domain that also appear in the range (for example, William). You will find that this is quite common for numerical relations.

Worked example 3.1

Snails keep eating Eva's tomato plant, so she decided to watch them for an hour one evening and find how far they travelled towards the plant. Her results are shown in the table. The initial distance from the tomato plant is x cm, and the distance after one hour is y cm.

x	321	540	282	236	82	123	211	211
y	274	523	198	187	30	156	145	209

Represent this information by a set of ordered pairs.

Solution

The set of initial distances, x cm, is the domain. The set of distances after one hour, y cm, is the range.

We have used ordered pairs before to define the x- and y-coordinates of points in a Cartesian graph. The order of the pair of numbers, say (3, 5), is important. The first number, 3, is in the domain, whereas the second, 5, is in the range.

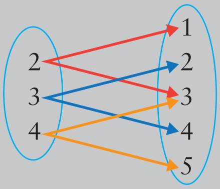
The set of ordered pairs for Eva's data is:

{(321, 274), (540, 523), (282, 198), (236, 187), (82, 30), (123, 156), (211, 145), (211, 209)}

Looking back, we see that for the snails there is no obvious rule connecting x to y, except that $y < x$ in most cases.

Worked example 3.2

Look at the mapping diagram.



a Identify the domain and range.

b Translate the information into a set of ordered pairs.

c Find a rule that could connect elements in the domain to those in the range.

d Follow the rule to suggest four additions to the domain and range in the form of ordered pairs.

e Use the ordered pairs as points and plot them on Cartesian axes.

Solution

Understand the problem

The arrows in the mapping diagram point from the elements in the domain to those in the range. Each arrow represents a connection, so there should be as many ordered pairs as there are arrows.

Make a plan

Identify the domain and range. To find a possible rule, pick one point in the domain and explain its connection to the points in the range. Then check the rule to see if it applies to the other points in the domain.

Carry out the plan

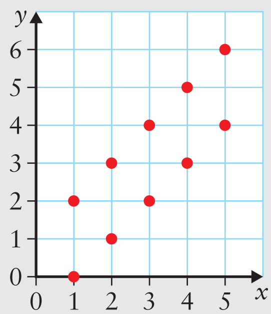
a The domain is the set $\{2, 3, 4\}$. The range is the set $\{1, 2, 3, 4, 5\}$

b The set of ordered pairs is $\{(2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 5)\}$

c 2 connects to 1 and 3, which are its nearest integer neighbours.
This works also for the connections from 3 and 4 in the domain.
So, the rule is that $x \to x \pm 1$

d If we include 1 and 5 in the domain and add 0 and 6 to the range then we obtain the additional pairs: (1, 0), (1, 2), (5, 4), and (5, 6)

e



Reflect

Is there enough evidence in the mapping diagram to be sure that you have the correct rule?

Would a graph of just the original pairs help you find the rule?

Might you have a relation in which there is no obvious rule?

Communication skills

Hint Q1a

You may need to look up some of these books.

Practice questions 3.1.1

1 Use the relation 'is the author of' to link the elements of the domain:

{Louisa May Alcott, Jane Austin, Agatha Christie, Charles Dickens, Victor Hugo, Jack London, J.K. Rowling}

with the range:

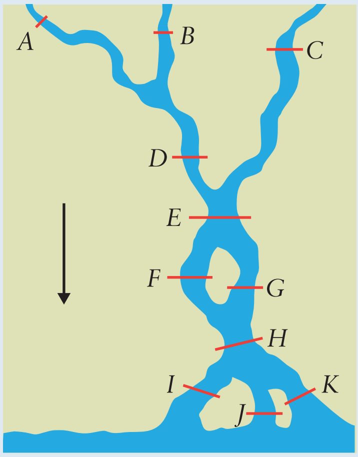
{A Christmas Carol, A Tale of Two Cities, Emma, Harry Potter and the Goblet of Fire, Les Misérables, Little Women, Murder on the Orient Express, Pride and Prejudice, The Call of the Wild, The Cuckoo's Calling, White Fang}

a Fill in a copy of the table.

b Draw a mapping diagram.

is the author of	A Christmas Carol	A Tale of Two Cities	Emma	Harry Potter and the Goblet of Fire	Les Misérables	Little Women	Murder on the Orient Express	Pride and Prejudice	The Call of the Wild	The Cuckoo's Calling	White Fang
Louisa May Alcott											
Jane Austin											
Agatha Christie											
Charles Dickens											
Victor Hugo											
Jack London											
J.K. Rowling											

2 The picture shows a river estuary with various observation points labelled A to K. The arrow shows the general direction of flow of the river.



Logs float unimpeded down the river towards the sea and are observed passing the various observation points.

a If a log is observed at a particular point, which other points does it definitely pass, either before or after the observation point?

b Create a table to show the connections.

3 For each of the following mapping diagrams:

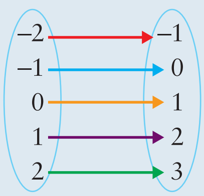
i identify the domain and range

ii write all the connections in the relation as a set of ordered pairs

iii where possible, suggest a rule for the relation

iv if you find a rule, suggest four further ordered pairs that would extend the domain and range while respecting the rule.

a



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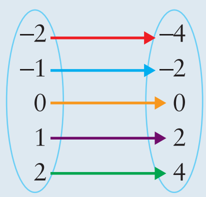

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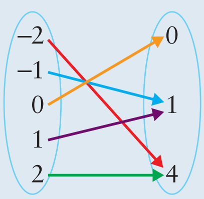
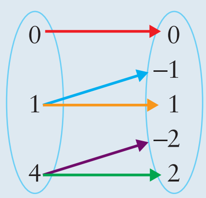


d



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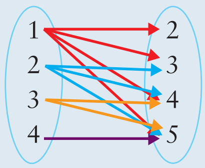

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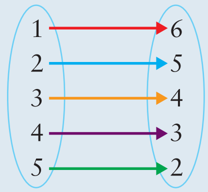

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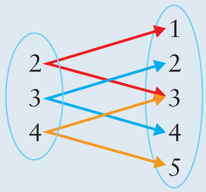


4 Draw a mapping diagram for each of the following sets of ordered pairs. In each case, plot the points corresponding to the ordered pairs on Cartesian axes and suggest a rule for the relation from the domain to the range. Add two more points, to each graph, that satisfy the same rule.

a $\{(3, 1), (6, 2), (9, 3), (12, 4)\}$

b $\{(1, 1), (2, 8), (3, 27), (4, 64)\}$

c $\{(0, 0), (1, -1), (2, -2), (3, -3)\}$

d $\{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

e $\{(0, 9), (1, 4), (2, 1), (3, 0), (4, 1), (5, 4), (6, 9)\}$

3.1.2 One-to-one relations

Explore 3.2

A computer has been set up to assign you an ID code made up of the first three letters of your family name followed by the first two letters of your given name. For example, the code for the name **AR**IANA **GRA**NDE would be **GRAAR**.

Would your ID code be unique in your class; in your extended family; in your school; in the entire country?

What could be added to the code to make sure it is unique?

You might need to remember a list for a test. There are some well-known techniques and here are two examples.

A memory aid

The common French verbs that conjugate with être in the passé composé (past perfect tense) are

Monter, **R**etourner, **S**ortir, **V**enir, **A**ller, **N**aître, **D**escendre, **E**ntrer, **R**ester, **T**omber, **R**entrer, **A**rriver, **M**ourir, **P**artir

In this order, the initial letters form 'Mrs Vandertramp', which is an invented name, but one that is quite easy to remember when it has been repeated a few times.

A memory palace

The noble gases are helium (He), neon (Ne), argon (Ar), krypton (Kr), xenon (Xe) and radon (Rn).

To remember them, you could attach post-it notes to various objects in your room (it does not have to be a palace) and get used to seeing them. When you want to remember them, close your eyes for a moment and think of your room, looking at each object to recall the names of the gases.

The first of these examples uses a **one-to-one relation**. Each verb has a single letter in Mrs Vandertramp and each letter in Mrs Vandertramp has a single verb.

In the memory palace, each noble gas has a single object in the room to identify it, but there are objects in the room that do not have labels. To make the second example one-to-one you would have to restrict the objects in the room to just those with labels.

Look at the picture on the opening page of this chapter. Each column on the right is connected to one column on the left. As you walk through, you do not need to count the columns to know that there are as many on the right as there are on the left. The columns are in a one-to-one relation. This is one of the most fundamental concepts of mathematics, and is the basis of our ability to count. If you have ever counted on your fingers, you have used a one-to-one relation between your fingers and the objects you are counting.

For a one-to-one relation there must be the same number of elements in the domain as in the range. This sounds easy to check, but there is the possibility that both sets are infinite. There must also be a rule that links an element in the domain to an element in the range but does not link either to more than one element.

Worked example 3.3

Determine whether or not there is a one-to-one relation:

a between the people in your class and their birthdays

b between a domain of the first ten positive whole numbers and a range of how many factors each number has

c between the number of sheep in a field and the total number of legs of the sheep in the field.

Solution

a Everyone in the class has a birthday. For this to be a one-to-one relation we have to apply the condition that no two people in the class can have the same birthday.

If that condition does not apply then it is not a one-to-one relation.

b All prime numbers have exactly two factors, so 2, 3, 5 and 7 in the domain will all map onto 2 in the range. Therefore, it cannot be a one-to-one relation.

c In this case there is a one-to-one relation. For example, with 10 sheep in the field there would be 40 legs, and if there were 40 legs there would be 10 sheep.

Connections

Factors: can you find two other counter-examples in addition to the prime numbers?

Practice questions 3.1.2

1 Identify which of the following situations involve a one-to-one relation.

a Everyone in a room is seated and there are no spare chairs.

b It is difficult to count animals in a field if they keep moving, so every day a farmer makes sure that no cows have wandered off by checking that every milking bay is occupied at milking time.

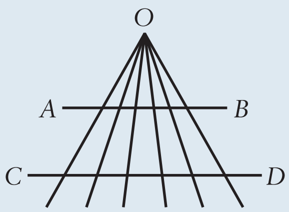
c The players in each of two football teams (before anyone is sent off).

d The letters A, B, C and the different arrangements that can be made from the letters A, B, C.

e Authors and the books written by the authors.

f The elements of the periodic table and their symbols.

g The points where the segment AB is cut by the rays coming from O and those where CD is cut by the same rays.



2 Which of the mapping diagrams in question 3 of Practice questions 3.1.1 represent one-to-one relations?

3 Which of the sets of ordered pairs in question 4 of Practice questions 3.1.1 represent one-to-one relations?

3.2 Functions

3.2.1 Definition of a function

Explore 3.3

$(-2)^2 = 4$, so would it be correct to say that -2 is the square root of 4?

$(-2)^3 = -8$, so is -2 the cube root of -8?

Ask similar questions for other integer powers, $(-2)^n$.

How would you define the nth root?

The distinction between **one** and **many** is important when describing functions. ‘Many’ in this context means more than one.

A function can be thought of as a process with an input and an output. Two different inputs could give the same output, but with a function you cannot obtain different outputs from the same input.

As it is a type of relation, we can use an arrow to show the link:
input → output

For example:

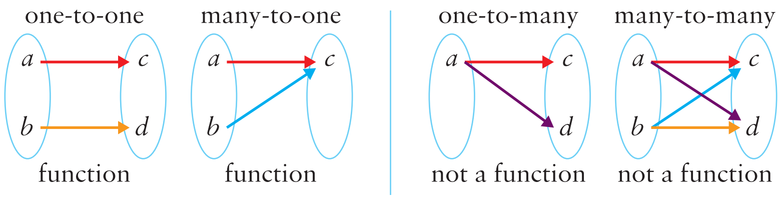
- **Add 5** is a function: $3 \rightarrow 8$ and $-3 \rightarrow 2$. There just one output for each input.
- **Square** is a function: $3 \rightarrow 9$ and $-3 \rightarrow 9$. In this case two different inputs give the same output, but there is still just one output to each input.

Functions are so common in mathematics that a relation that is not a function usually needs a longer description. A simple example would be **'is a factor of'**. This definition would take an input of 12 and give six numbers in the output: $12 \rightarrow 1, 2, 3, 4, 6, 12$

With six possible outputs from one input, this cannot be a function.

Remember that a function gives a unique output for each input.

We define a function as a **one-to-one** or **many-to-one** relation.



In a mapping diagram, if there is only one arrow leaving each element of the domain then it fulfils the one-to-one or many-to-one condition. It represents a function because there is no doubt about where that element is mapped to in the range.

If there is at least one member of the domain that has two or more arrows leaving it, then the relation is not a function.

Worked example 3.4

Consider the following sets of ordered pairs defining a relation from the first number in the ordered pair to the second number.

$$A = \{(0, 2), (1, 3), (2, 3), (3, 3), (4, -3)\}$$
$$B = \{(0, -3), (0, 2), (2, 1), (2, 3), (3, 1)\}$$

Do A and B represent functions?

Solution

Understand the problem

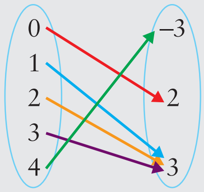
In each of the ordered pairs, the first number is in the domain and the second in the range. The number of connections between numbers is what determines whether or not this relation is a function.

Make a plan

We can draw a mapping diagram based on the ordered pairs to show the connections clearly.

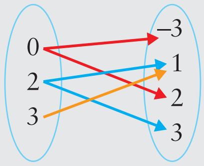
Carry out the plan

A has the domain $\{0, 1, 2, 3, 4\}$ and the range $\{-3, 2, 3\}$.



For A, each number in the domain has a single arrow leaving it, so relation A is a function. Because 3 in the range has many arrows arriving at it, it is a many-to-one function.

B has the domain $\{0, 2, 3\}$ and the range $\{-3, 1, 2, 3\}$.



Both 0 and 2 in the domain have many (more than one) arrows coming from them, so this is not a function. In the range, 1 has many arrows arriving, so it is a many-to-many relation.

Look back

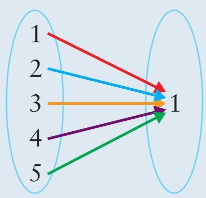
The key to identifying a function is knowing that a number in the domain should have only one corresponding number in the range. In practical terms, this usually involves a rule for which there is only one output for each input. For example, the rule might be to square each element in the domain.

Practice questions 3.2.1

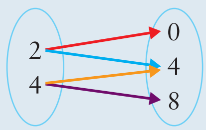
1 Which mapping diagram in question 3 of Practice questions 3.1.1 represents a many-to-one function?

2 For each of the following relations, decide if it is a one-to-one function, a many-to-one function, or not a function at all.

a



b



c The set of ordered pairs $\{(1, 5), (2, 9), (3, 13), (4, 17), (5, 21)\}$

d Any integer mapped to a number twice its size

e Any integer mapped to its square

f Any integer mapped to its cube

g Any square mapped to the integers that could be squared to obtain it

h The example of the British royal family from the start of the chapter, with the domain and range swapped and using the relation 'is the child of'

i The planets of our solar system mapped to their moons

j Any positive integer mapped to its approximate value when rounded to the nearest multiple of 10

k A number written to the nearest 10 mapped to the original number

3 32 chocolates are shared equally between a number of people, with some chocolates remaining. Is the relation connecting the number remaining to the number of people a function?

4 Consider a relation that maps x onto a number y that is greater than x.

a Write a list of five numbers greater than 10 and give some possible values of y when $x = 10$

b If you knew that $y = 13$, would you know that $x = 10$?

c Is x a function of y?

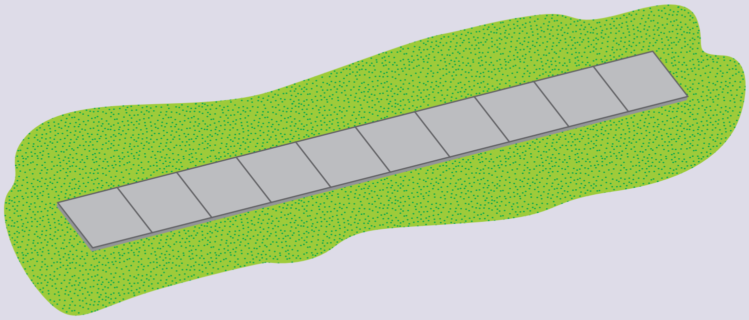
d Is y a function of x?

e Would your answers to parts c and d be different if x mapped onto a number y that is the smallest integer greater than x?

3.2.2 Beyond integer domains

Explore 3.4

To finish a path of 80 cm width there is a choice between pouring concrete or laying paving slabs. The slabs are squares measuring 80 cm by 80 cm. If they do not fit exactly into the length of the path, then a whole slab will have to be cut to length. The concrete can be mixed to fill a path of any length.



Pouring the concrete will cost €12 per metre length of path. For example, pouring an 8-m path would cost €96.

Paving slabs cost €9.25 each. For an 8-m path, the cost would be €92.50, with 10 slabs fitting exactly and no waste from cutting a slab.

Calculate the costs of each method for a 9-m path.

On a single set of axes, draw graphs to show the costs for each method for whole number path lengths between 5 m and 12 m.

What is the difference between the two graphs?

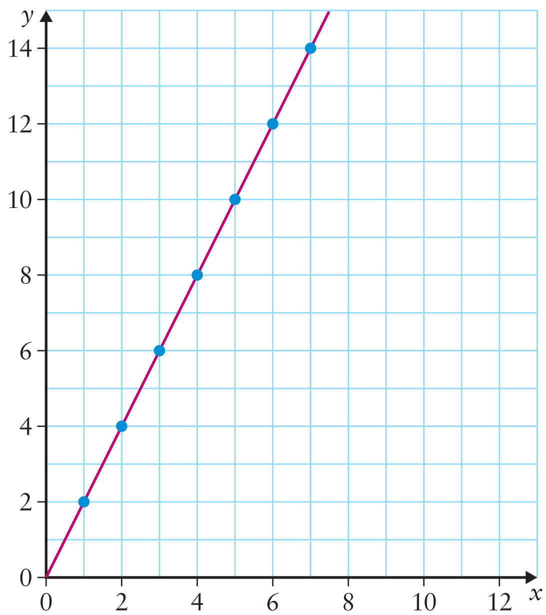
Reflect

Look back at your solutions to the one-to-one relations in questions 3 and 4 of Practice questions 3.1.1, particularly the values you added when there was a clear rule. Also look at the graphs you drew. Was it tempting to join the dots? Would it have been justified?

In the numerical examples so far in this chapter, the domain and range have been elements from the set of integers.

Consider the set of ordered pairs $\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$.

The rule for the relation is apparently 'double the number'. So, we could enlarge the domain with (6, 12) and (7, 14). But we could also include points between those already given, for example $\left(\frac{1}{2}, 1\right)$ or $\left(\frac{5}{4}, \frac{5}{2}\right)$ which still obey the same rule. Thus the domain is no longer restricted to integers.



When all the points are plotted, a straight line can be drawn through them with the equation $y = 2x$. Any point on this line could be added to the domain and range without altering the rule.

In this way, we can extend the idea of relations between sets of integers to relations between sets of real numbers. The idea of one-to-one starts to get more complicated as there are an infinite number of mappings even within a small interval.

Reflect

Find the relations from questions 3 and 4 of Practice questions 3.1.1 that produce points lying on straight lines when graphed. Look for a pattern that applies to their rules.

Self-management skills

Worked example 3.5

During the coming month, a student will be commuting frequently to study at a distant university. The journey is by train and these are the ticket price options:

Present this information as a graph.

What recommendations would you give the student?

Solution

Understand the problem

The decision facing the student depends on the number of return journeys in the month. To make a recommendation we need to show how the cost of each option varies as the number of journeys changes. The number of journeys is the independent variable and belongs to the domain. The total cost of one-day returns depends directly on the number of return journeys. The season ticket is a fixed cost that does not vary with the number of journeys. The railcard is in part a fixed cost, and in part dependent on the number of journeys. The three different costs rely on three different functions and each belongs to the range of its own function.

A 40% reduction leaves 60% to pay.

Make a plan

We can draw a graph for each function to show which is cheapest for different numbers of journeys.

Typically, a graph uses x as the independent variable. There are three different formulae, so each has its own graph and range. To plot them on a graph, we calculate the cost €y in terms of the number of journeys, x.

We can see, apart from the season ticket, that the cost will rise in direct proportion to the number of journeys, so it is reasonable to assume a linear relationship. Only two points are needed to position a straight line, so two examples from each type of ticket will be enough for the graph.

Carry out the plan

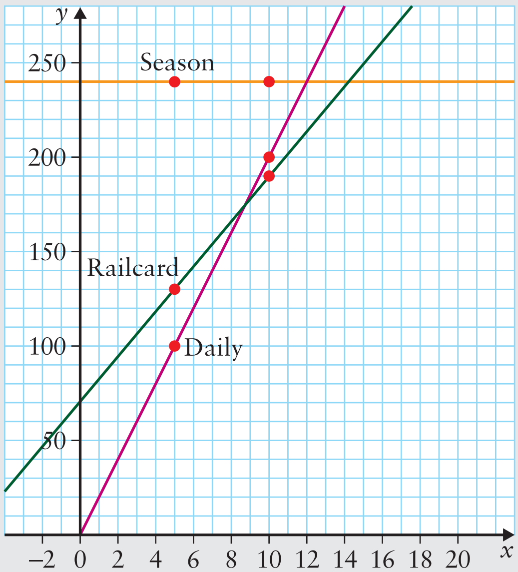
For five return journeys ($x = 5$) the cost in euros will be:

- One-day return: $y = 20 \times 5 =$ €100, giving the ordered pair (5, 100)
- Monthly season ticket: $y =$ €240, giving the ordered pair (5, 240)
- Monthly railcard: $y = 70 + 0.6 \times 20 \times 5 =$ €130, giving the ordered pair (5, 130)

For 10 return journeys ($x = 10$) the cost in euros will be:

- One-day return: $y = 20 \times 10 = €200$, giving the ordered pair (10, 200)
- Monthly season ticket: $y = €240$, giving the ordered pair (10, 240)
- Monthly railcard: $y = 70 + 0.6 \times 20 \times 10 = €190$, giving the ordered pair (10, 190)

Plotting the points for each ticket and drawing the lines gives us a way to compare the costs.



From the graph, up to 8 return journeys in the month would be cheapest with one-day returns, between 9 and 14 would be cheaper with the railcard and 15 and above would be cheaper with the season ticket.

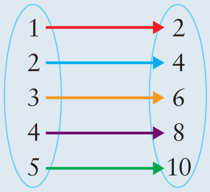
Look back

While it is convenient to plot the lines continuously, the domain is restricted to positive whole values as it is not possible to make fractional or negative return journeys.

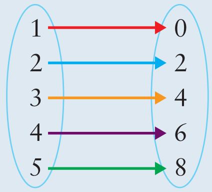
Practice questions 3.2.2

1 For each of the mapping diagrams, identify the rule and plot the ordered pairs on a coordinate graph. Then join the points to extend the domain to include all real numbers within the limits of the original domain.

a



b



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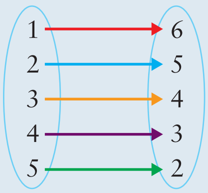

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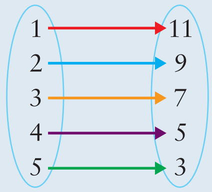


2 For each part in question 1, find the value in the range that corresponds to 1.1 in the domain.

3 You work in a shop which has a wide range of items for sale between €49.99 and €499.99.

The owner decides to clear a lot of stock to reorganise the layout of the shop. She therefore decides to have a sale to encourage people to purchase the stock. She is undecided on what offer to make and asks you to create a single sheet giving a visual comparison of the sale price against the original price for each of the following options:

A: €30 off every price

B: a 15% reduction in all prices

C: a 10% reduction and then €10 off the resulting price

Represent the three offers on a single set of axes showing graphs for the reduced costs against the original costs for any price between €49.99 and €499.99.

4 At the airport, you can change your money from dollars into euros. The service costs $5, and for every additional dollar you get €0.85.

a Is the relation between the number of dollars and the number of euros a function?

b How many euros would you get for $100?

c How many euros would you get for $200?

d Use the results to plot points on a graph and draw a line through them to show the relation.

e Use your graph to determine how many dollars it would cost to obtain €50.

3.2.3 Function notation

Explore 3.5

The value of a function is defined as the original number subtracted from its cube.

Consider, but don't necessarily answer, the following questions:

a What is the value of the function when the original number is 0?

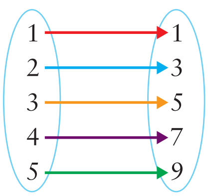
b What is the value of the function when the original number is −2?

c Which original number could give 24 as the value of the function?

d Which original numbers would be trebled by the function?

Devise a way of asking these questions so that they are much shorter, using symbols rather than words wherever possible. You might like to create your own notation.

A function converts the positive integers into the odd numbers. The first few terms are shown in the mapping diagram, but there are infinitely many terms, so a mapping diagram cannot show the full domain and range. A set of ordered pairs would have the same limitation.



What is needed is a way to express the rule: doubling the number in the domain and subtracting 1.

If n is any number in the domain (the set of positive integers) then the rule would turn it into $2n - 1$. This is a function, as it is not possible to obtain two different answers in the range for one value of n in the domain.

We will call the function f.

We can define the function as:

$f(n) = 2n - 1$

We read this as 'f of n equals two n minus one'.

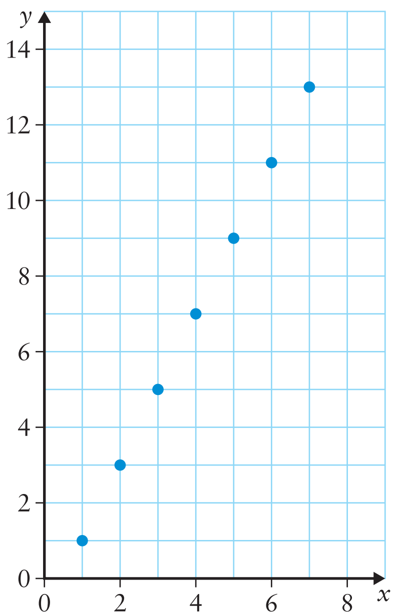
We can replace n with positive integers, so $f(2) = 3$ and $f(100) = 199$, but we can also replace n with another letter: $f(x) = 2x - 1$

$f(n) = 2n - 1$ and $f(x) = 2x - 1$ each describe the function f completely.

To draw a Cartesian graph, we need a relation between x- and y-coordinates to decide whether a point in the plane is part of the graph.

We put $y = f(x)$

The domain for this example could be that x is a positive integer, $x \in \mathbb{Z}^+$



Unlike the function notation, the graph is limited in the same way as the mapping diagram. It can only show some relations. The positive integers are infinite, but our graph is finite.

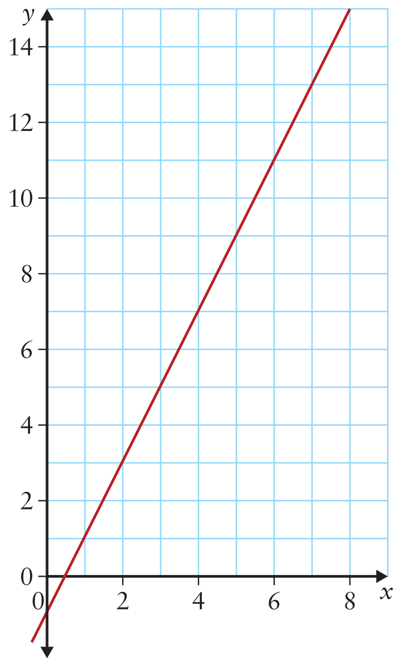
Fact

There is another form of notation that is often used to define functions, which imitates a mapping diagram.

$f: n \rightarrow 2n - 1$

We read this as 'f maps n onto two n minus one'.

If we change the domain to all real numbers, the graph has an advantage as it represents an infinite number of values along the line in a way that the mapping diagram cannot.



We describe this as the graph of $y = f(x)$, where $f(x) = 2x - 1$, defined on the domain $x \in \mathbb{R}$.

This is a long way of writing what we would call the graph of $y = 2x - 1$, but the notation allows much greater flexibility in use.

Communication skills

f is the most common symbol used to name a function, but there are many other names used. We often use $f(x)$ and $g(x)$ when comparing two functions and you will encounter common mathematical functions with abbreviated names, such as $\sin(\theta)$ and $\log(x)$.

Worked example 3.6

A function is defined on the set of real numbers by $f(x) = x^3 - x$.

Find the value of:

a $f(1)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(0)$ **e** $f(10)$

Solution

a $f(1) = 1^3 - 1 = 1 - 1 = 0$

b $f(2) = 2^3 - 2 = 8 - 2 = 6$

c $f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$

d $f(0) = 0^3 - 0 = 0 - 0 = 0$

e $f(10) = 10^3 - 10 = 1000 - 10 = 990$

Worked example 3.7

Maddy is comparing two deals from rival electricity suppliers.

Deal A: A standing charge of €30 for every two-month period and a price per unit of €0.10

Deal B: A standing charge of €10 for every two-month period and a price per unit of €0.20

Which would be the better deal for a household that uses 300 units of electricity in a two-month period? Describe both deals in function notation. Illustrate your answer with a graph.

Solution

Understand the problem

The standing charge is paid even if no electricity is used.

The rate of increase in price for Deal B is greater than that for Deal A, so it will give a steeper graph.

Which of the two deals is likely to be cheaper if you use very few units, or if you use a lot?

Make a plan

To put these deals into function notation, a variable is needed and each function has to be named. The variable is the number of units used and combines with the cost per unit, but has no effect on the standing charge.

For a graph, it is a good idea to name the variable x. The functions must produce numbers. They will describe the number of euros in each case.

The question about 300 units gives an indication of the size of the domain and therefore of the x axis.

Carry out the plan

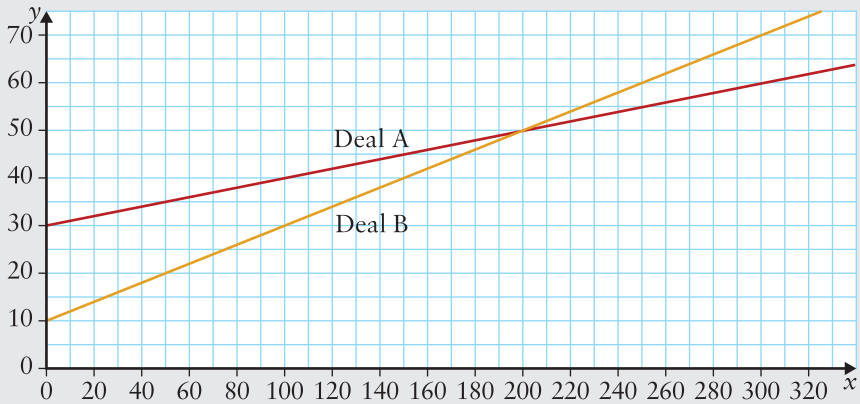
Let x be the number of units used over two months.

Let $A(x)$ be the cost of x units under Deal A in euros.

Let $B(x)$ be the cost of x units under Deal B in euros.

Then $A(x) = 30 + 0.1x$ and $B(x) = 10 + 0.2x$

The graphs of $y = A(x)$ and $y = B(x)$ are shown.



The lines cross at $x = 200$, so any consumption above 200 units per two-month period would cost less with Deal A than Deal B.

Evaluating at $x = 300$:

$A(300) = 30 + 0.1 \times 300 = 60$

$B(300) = 10 + 0.2 \times 300 = 70$

Practice questions 3.2.3

1 Evaluate each of the following functions at the given values from the domain.

a $f(x) = 4x$, find $f(3)$ and $f(-1)$

b $f(x) = x - 2$, find $f(4)$ and $f(2)$

c $f(x) = 4 + x$, find $f(10)$ and $f(-10)$

d $f(x) = x$ rounded to the nearest integer, find $f(2.2)$, $f(1.9)$, $f(-1.3)$

e $f(x) = \frac{12}{x}$, find $f(4)$, $f(2)$, $f(12)$

f $f(x) = 4x^2$, find $f(0)$, $f(2)$, $f(-2)$

2 For each function in question 1, find a possible value of x such that $f(x) = 4$

3 Which functions in question 1 are many-to-one?

Challenge Q4

4 Which of the functions in question 1 cannot have the complete domain of all integers?

5 The cooking time for a joint of beef depends on taste preferences and on the size of the joint. The oven is pre-heated to 180°C and then:

- for rare (lightly cooked) meat, allow 20 minutes for every 450 grams plus an additional 20 minutes
- for medium cooked meat, allow 25 minutes for every 450 grams plus an additional 25 minutes
- for well-done meat, allow 30 minutes for every 450 grams plus an additional 30 minutes.

a Calculate the time taken to cook an 800 gram joint for each of rare, medium and well-done. Keep a note of your calculations as a model for part b.

b Find three functions, $r(x)$, $m(x)$ and $w(x)$, for the cooking time of rare, medium and well done joints respectively, given that the weight of the joint is x grams.

c Use technology to plot the three functions for a sensible domain.

d Two beef joints are put in a large pre-heated oven and taken out at the same time. One joint weighs 1 kg and is to be served rare. The other is to be served well-done. What is the weight of the well-done joint?

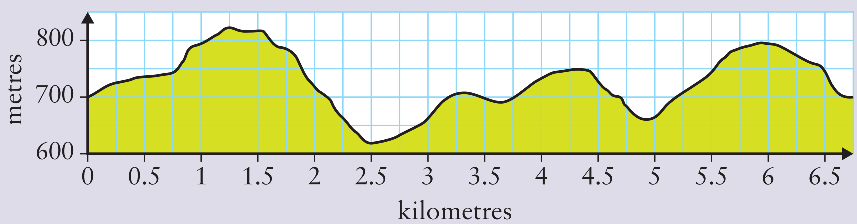
Hint Q5c

Would you cook 5 grams of meat in this way?

3.2.4 Which graphs represent functions?

Explore 3.6

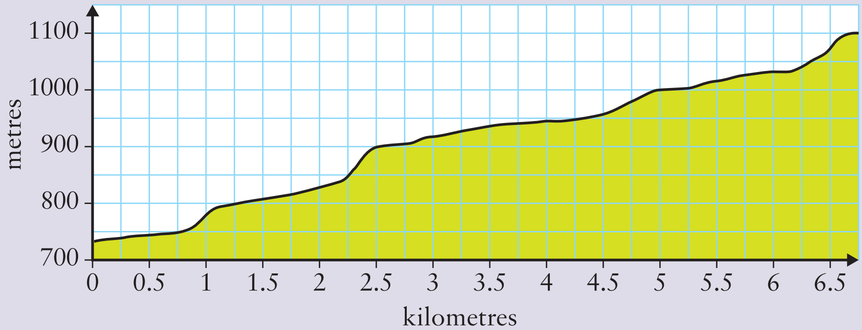
You are planning a walk in a mountainous area. Before you set off, you print an outline map of the route showing the variation in height above sea level.



If you had been walking for 3 km, could you use the map to work out your height above sea level? Would you be able to work out the height at any distance along the walk?

If you knew you were 750 m above sea level, could you use the map to work out how far you had walked? Are there any heights above sea level for which you would know how far you had walked?

Would your answers be different if you were walking to the top of a hill with this map?



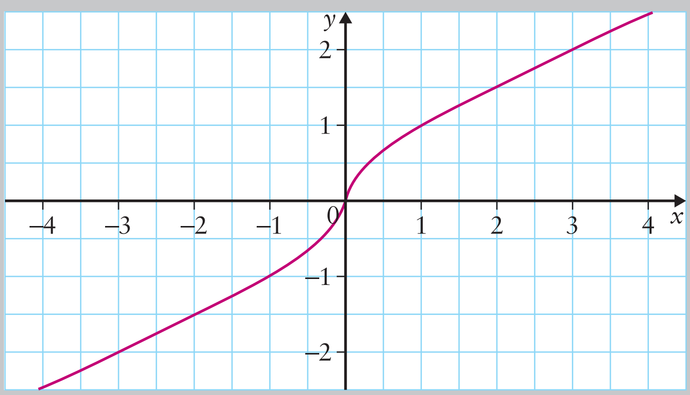
What is it about the nature of the maps that makes a difference?

So far, we have looked at linear functions and their graphs. Unless the line is vertical (with equation $x = c$) it is fairly obvious that a straight-line graph represents y as a function of x. But you will use functions to describe many different shaped graphs. It is possible to identify whether a relation is a function from its graph.

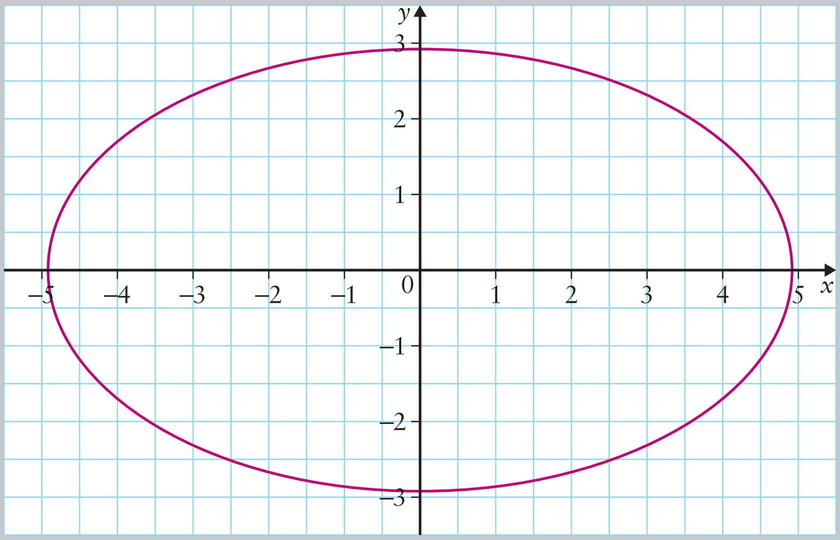
Worked example 3.8

Which of these graphs represents y as a function of x?

a



b



Solution

Understand the problem

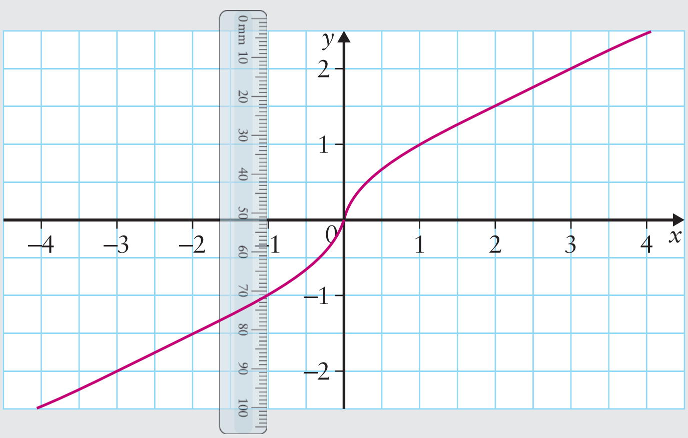
If a graph represents a function, it should be possible to choose any value of x and use the graph to find a unique corresponding value of $y = f(x)$

Make a plan

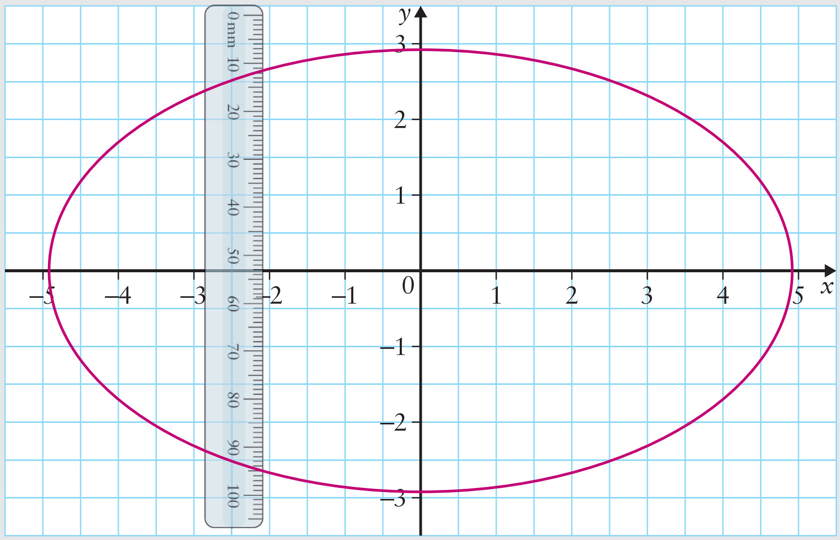
To test for a unique value, we can represent possible values of x (those in the domain and represented on the graph) by vertical lines. We should check at how many points the vertical lines meet the graph. A ruler can be a good instrument for such a check.

Carry out the plan

a As a ruler in a vertical position is moved from left to right across the graph, we can see that it meets the graph in at most one point at a time. Therefore this graph represents a function



b A ruler in a vertical position meets the graph at two points almost everywhere in the domain that is shown. Therefore this graph does not represent a function.



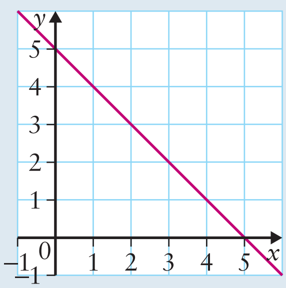
Look back

It seems clear that the relation represented in graph b is not a function, but the relation in graph a is a function, as it does meet the condition of a unique value each time. This method is known as the **vertical line test**.

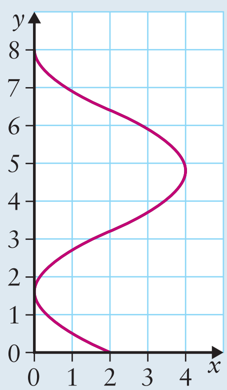
Practice questions 3.2.4

1 Use the vertical line test to determine which of the following graphs represent y as a function of x over the domain that is shown.

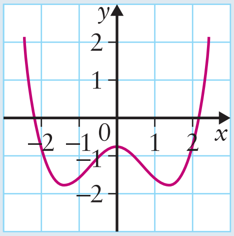
a



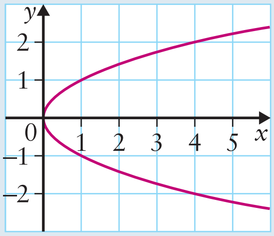
b



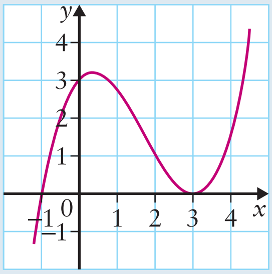
c



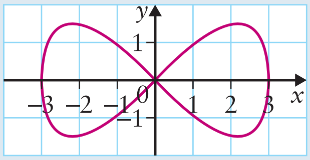
d



e



f



2 Despite the change to the metric system in 1971, the UK still makes regular use of imperial measures. One example is the fuel consumption of a car, which is usually given in miles per gallon (mpg) in the UK. A far more common measure worldwide is litres per 100 km.

- 1 gallon = 4.546 litres
- 1 mile = 1.609 km

a Apart from the size of the units, what is the fundamental difference between the two measures?

b Convert 50 mpg to litres per 100 km.

c Use your calculation from part b as a model to find the function $L(x)$ that will express x mpg in litres per 100 km. Give your answer in its simplest form.

d Find the function $M(x)$ that will express x litres per 100 km in mpg.

e Check that your functions work with 50 mpg and its equivalent in litres per 100 km

f How do you explain the similarity of the two functions?

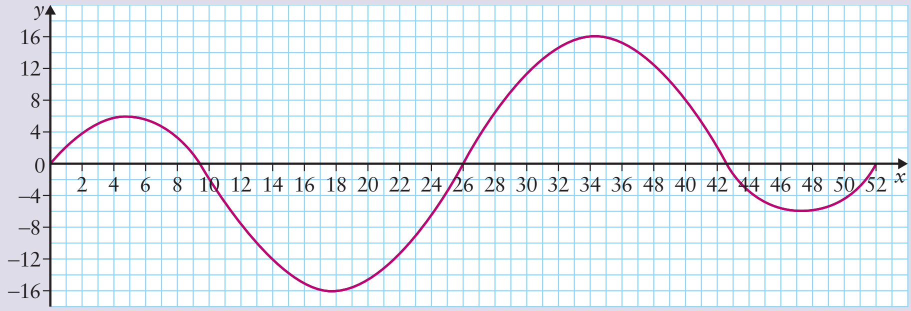
g Use technology to draw the graph of $y = L(x)$ for a suitable domain and range. Compare it with the graph of $y = M(x)$

h A car that is parked with the engine switched off is consuming fuel at a rate of 0 litres per 100 km. Can you express that in mpg?

3.2.5 Inverse functions

Explore 3.7

You might expect the Sun to be at its highest in the sky at the same time every day, but it is not quite that simple. For that to happen, the Earth would have to orbit the Sun in a circle and not tilt on its axis. But in fact, the Earth's orbit is an ellipse and the tilt is about 23.5°. The result is that time measured by a sundial is sometimes ahead of the clock and sometimes behind by up to 16 minutes during the course of a year. The graph (called the Equation of Time) shows the variation in minutes by week of the year. For example, the sundial and clock are aligned 9 weeks and four days into the year (8th March, if not a leap year).



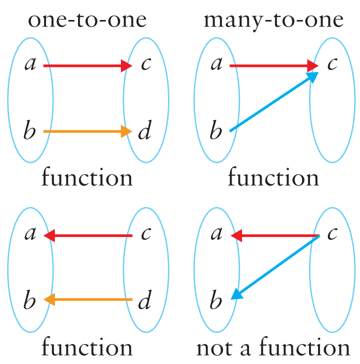
Provided you knew the date, could you use the graph to determine when, either side of a clock's midday, the Sun would be highest in the sky?

If you recorded the time between sundial midday and clock midday, could you use the graph to determine what day of the year it was? Might you need some additional information?

One important question about functions is whether or not the process can be reversed.

Consider the periodic table. We need to decode the abbreviations. If H were used to represent both hydrogen and helium, there would be a problem, which is why He is used for helium. You have seen earlier that the symbols for the elements are in a one-to-one relation with the names of the elements. That becomes important when the relation between sets must work both ways.

To reverse the process of a function, the range and domain swap places and the arrows in a mapping diagram point in the opposite direction. If the rule to achieve this is also a function, it is called the **inverse function**. We can try this with the functions we saw earlier.



Only the one-to-one function has an inverse.

Fact

Inverse functions can be found only for functions that are one-to-one.

We use the notation f^{-1} to represent the inverse of f.

Worked example 3.9

The following rules define functions on the domain of integers. Find the inverse functions of each, if they exist.

a Multiply by 3

b Square

c Add 50

d Subtract from 100

Solution

In each case, the existence of an inverse function depends on whether the original function is one-to-one. Solving simple equations can give the impression that any step can be undone, but that is based on assumptions about the domain that should be examined in more detail.

As we are told that these are functions, we need to work out whether many (more than one) elements of the domain map to the same element of the range.

a Multiplication is one-to-one (there is only one number that is three times as big as another) and is undone by division, so the inverse function has the rule 'divide by 3'.

b Squaring when the domain includes negative numbers is many-to-one. For example, $4^2 = 16$ and $(-4)^2 = 16$. So, the function is not one-to-one and there is no inverse.

c Addition is one-to-one (there is only one number that is 50 more than another). The inverse function has the rule 'subtract 50'.

d Consider some examples:

$100 - 45 = 55$, $100 - 55 = 45$,

$100 - (-1) = 101$, $100 - 101 = -1$

It becomes clear that subtracting from 100 is reversed by the same process. This is called a **self-inverse**.

Fact

A function is a self-inverse if the inverse function has exactly the same definition as the original function.

Reflect

Reflections are examples of self-inverses. From a mathematical point of view, you are your reflection's reflection.

Consider the reciprocal function: $f(x) = \frac{1}{x}$

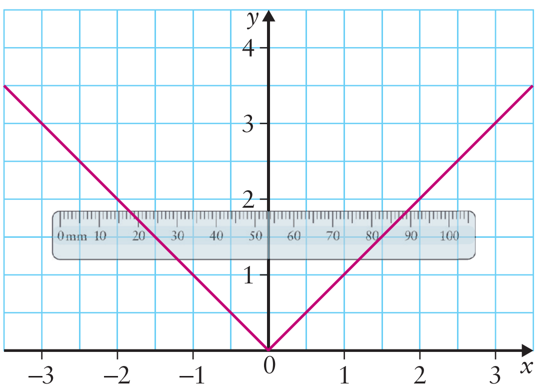
$f\left(\frac{2}{3}\right) = \frac{3}{2}$ and $f\left(\frac{3}{2}\right) = \frac{2}{3}$

f is a self-inverse: $f^{-1}(x) = \frac{1}{x} = f(x)$

Explore these examples: $g(x) = -x$, $h(x) = 1 - x$

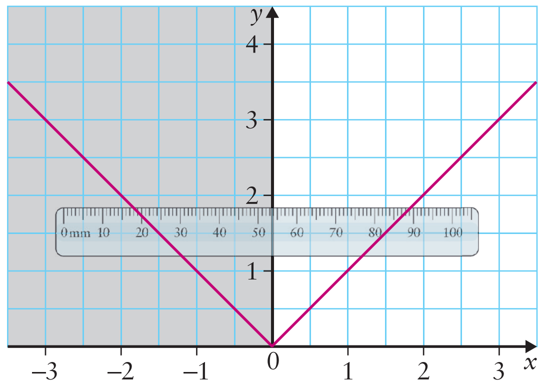
A graph can show y as a function of x. The inverse would be x as a function of y. Again, this is only possible if the original function is one-to-one.

Testing whether x is a function of y on a graph can be done in a very similar way to the test for y as function of x, this time using a **horizontal line test**.



The graph shows a relation that would pass the vertical line test, but not the horizontal line test. The ruler touches the graph at two points, so choosing a value of y would give two values of x and therefore x is not a function of y.

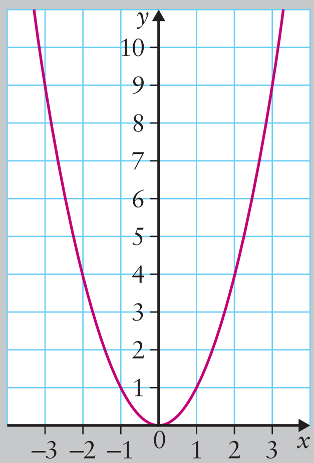
Sometimes an inverse function can be established by limiting the domain and range.



In this case only positive values of x and y are included. This means that x is a function of y.

Worked example 3.10

The graph, $y = f(x)$, of the function $f(x) = x^2$ is defined on the domain of all real numbers.
Explain why f has no inverse function.
What limitation could be made to be able to define the inverse function f^{-1}?



Solution

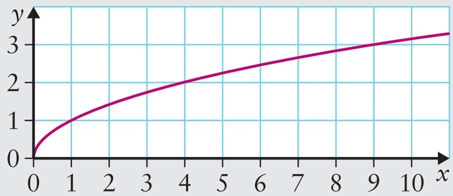
First we need to check that f is a function. That could be done with a vertical line test on the graph.

An inverse function would map $f(x)$ to x. In other words, x would be a function of y that can be checked with a horizontal line test.

A horizontal line test on the same graph reveals that x is not a function of y.

If the domain of f (therefore the range of f^{-1}) was restricted to just positive values, then the inverse function would be $f^{-1}(y) = \sqrt{y}$

Although it may seem confusing, it is important to note that $f^{-1}(y) = \sqrt{y}$ can be rewritten as $f^{-1}(x) = \sqrt{x}$, as x or y are only acting as go-betweens to describe what the function does. The advantage of swapping from y to x is that it allows us to draw a new graph showing $y = f^{-1}(x)$



Worked example 3.11

Given that the function $f(x) = 3 - 2x$ is one-to-one on the domain of real numbers, find a definition of the inverse function.

Solution

If $y = f(x)$, then y is a function of x. The inverse will be given by x as a function of y. So make x the subject of the equation.

$y = f(x) \Rightarrow x = f^{-1}(y)$

Let $y = f(x) = 3 - 2x$

Then $y + 2x = 3 \Rightarrow 2x = 3 - y \Rightarrow x = \dfrac{3-y}{2}$

So $f^{-1}(y) = \dfrac{3-y}{2}$

If needed, this can be restated as $f^{-1}(x) = \dfrac{3-x}{2}$

Worked example 3.11 is about a function involving two steps: multiplication by 2 and then subtraction from 3. It is therefore sufficiently complicated to benefit from a formal approach.

Simpler functions can often be dealt with using prior knowledge.

For example, given $f(x) = x - 2$, we can state that the opposite of subtracting 2 is adding 2 and thus $f^{-1}(x) = x + 2$

We should always check the domain of f^{-1}. If the domain of f were positive integers, then its range would be integers greater than or equal to -1, and that would therefore become the domain of f^{-1}.

Practice questions 3.2.5

1 Answer these questions for each of the following graphs.

i Does the graph pass the vertical line test?

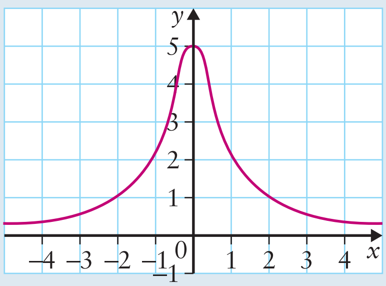
ii Does the graph represent y as a function of x?

iii Does the graph pass the horizontal line test?

iv If the graph represents y as a function of x, is it a one-to-one or a many-to-one function?

v Does the graph represent x as a function of y?

a



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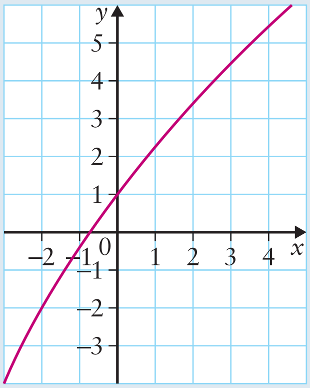

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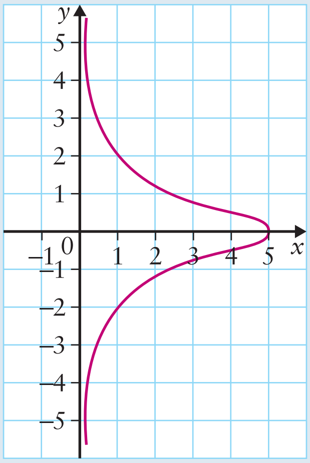
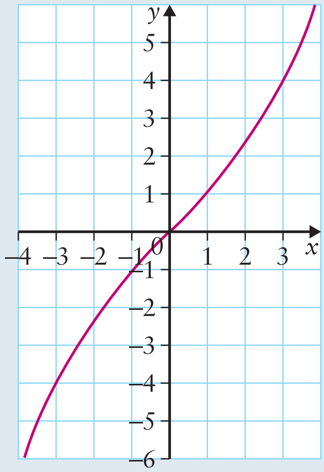


d



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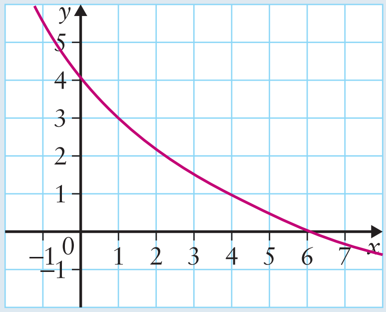

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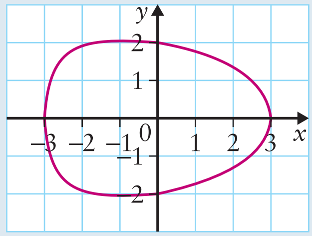


2 All the following functions are one-to-one and defined on the set of real numbers. Find the inverse function for each one, stating it in the form $f^{-1}(x)$.

a $f(x) = x + 1$ b $f(x) = x - 1$ c $f(x) = 3x$

d $f(x) = \frac{x}{5}$ e $f(x) = 2x - 1$ f $f(x) = 1 - x$

g $f(x) = 4 + 3x$

Challenge Q3

3 Given that $f(x) = x^4$ is defined on the set of integers, give examples to explain why the function does not have an inverse.

Challenge Q4

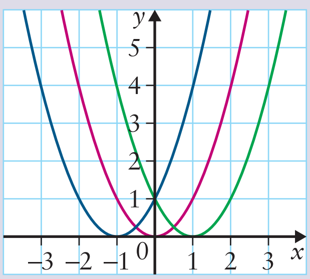
4 Given that $f(x) = x^2$ has an inverse function, give two examples of possible domains for f.

3.2.6 Transformations of graphs of functions

Explore 3.8

The graphs here are of $y = x^2$, $y = (x + 1)^2$ and $y = (x - 1)^2$

Use technology to check which is which.



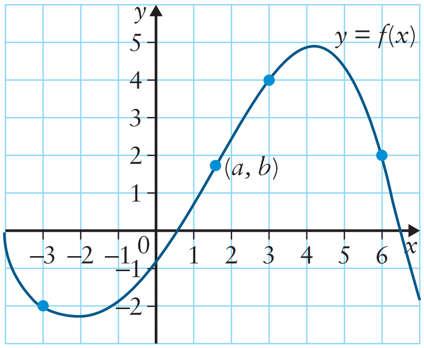
Continue to investigate graphs of the form $y = (x + a)^2$ using, for example, a constant controller, scrollbar (slider) or spin button to vary a.

What do you observe?

This graph shows $y = f(x)$.
For example, $b = f(a)$ since (a, b) is a point on the graph.

The definition of the function f is not important for what follows.

We can see that the graph passes through $(6, 2)$, $(3, 4)$ and $(-3, -2)$



We will now apply the function f, not to x, but to $3x$, and find the graph of $y = f(3x)$

This will transform the graph. The aim is to describe the **transformation**. We will do this by finding values in the domain, based on values in the range.

Using the point $(6, 2)$ as an example, 2 in the range comes from 6 in the domain (among other values, because the function is not one-to-one).

So we know $2 = f(6)$

What value of x would give $y = 2$ if we use $y = f(3x)$?

We would need $3x = 6$, so $x = 2$

It follows that $(2, 2)$ is a point on $y = f(3x)$ that corresponds to $(6, 2)$ on $y = f(x)$

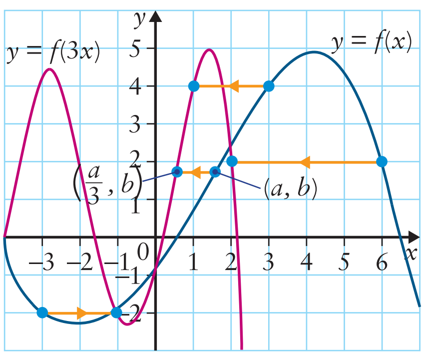
Follow the process again to see that:

- $(1, 4)$ on $y = f(3x)$ corresponds to $(3, 4)$ on $y = f(x)$
- $(-1, -2)$ on $y = f(3x)$ corresponds to $(-3, -2)$ on $y = f(x)$

Finally, the point (a, b) on $y = f(x)$ corresponds to $\left(\frac{a}{3}, b\right)$ on $y = f(3x)$

As (a, b) could be anywhere on $y = f(x)$ we have a general guide for where to plot points on $y = f(3x)$: for a given point on $y = f(x)$, the corresponding point with the same y-coordinate on $y = f(3x)$ will have an x coordinate $\frac{1}{3}$ of the original.

The result is that all points from the graph of $y = f(x)$ move closer to the y-axis to appear on the graph of $y = f(3x)$



Worked example 3.12

Sketch the graph of $2y = f(x)$ from the graph of $y = f(x)$ on the previous page.

Solution

This time the change is made in the range (from y to $2y$), not the domain, so we need to find where a value in the domain goes.

From the points on the graph, we see that $2 = f(6)$, $4 = f(3)$, $-2 = f(-3)$, $b = f(a)$

What value of y would come from $x = 6$ in the case of $2y = f(x)$?

Which point on the graph of $2y = f(x)$ corresponds to $(6, 2)$ on $y = f(x)$?

What does this and the transformations of the other identified points tell us about the new graph?

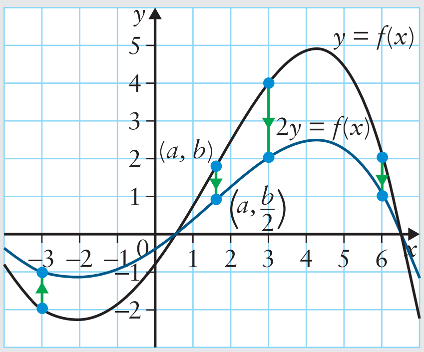
Given that $2 = f(6)$, and $2y = f(x)$, we need $2y = 2$, so $y = 1$

It follows that $(6, 1)$ is a point on $2y = f(x)$ that corresponds to $(6, 1)$ on $y = f(x)$.

Similarly:

- $(3, 2)$ on $2y = f(x)$ corresponds to $(3, 4)$ on $y = f(x)$
- $(-3, -1)$ on $2y = f(x)$ corresponds to $(-3, -2)$ on $y = f(x)$

(a, b) on $y = f(x)$ must become $\left(a, \frac{b}{2}\right)$ on $2y = f(x)$. From this, we see that all points from the graph of $y = f(x)$ move half of the way to the x-axis to appear on the graph of $2y = f(x)$



In general, we can think about these transformations in terms of compensation.

If the coefficient of x is doubled, then the value of x is halved to compensate and give the same total value for the term.

If 5 is added to x, then the value of x can be made smaller by 5 to compensate and give the same total value.

Connections

Inverse proportion: For ax to remain the same size when a is increased, x must decrease in inverse proportion. For example, $ax = 12$ with $a = 2$ gives $x = 6$, but with $a = 4$ it gives $x = 3$

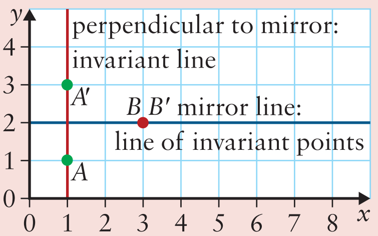
Similar statements can be made about the y-coordinates.

The transformations studied in this chapter are either **stretches**, **translations** or **reflections**.

Fact

Under a transformation:

- a **line of invariant points** is one where neither the line nor the points on the line change position
- an **invariant line** is one where the line does not change position but the points on the line might move to another position on the line.



For example, in the case of a reflection, the mirror line, $y = 2$, is a line of invariant points: B and its image B' remain in the same place.

Lines perpendicular to the mirror line, $x = 1$, are invariant lines: points on a perpendicular line, A and A', swap positions, but remain on and define the same line.

The transformation from $y = f(x)$ to $y = f(3x)$ is a stretch with scale factor $\frac{1}{3}$ parallel to the x-axis, with the y-axis as the line of invariant points.

The transformation from $y = f(x)$ to $2y = f(x)$ is a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis, with the x-axis as the line of invariant points.

Fact

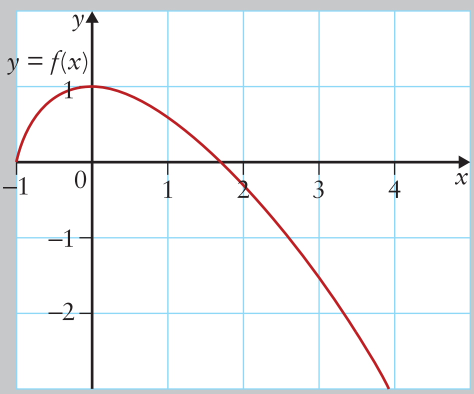
A one-way stretch has the effect of moving points perpendicularly to a fixed line. The new distance from the line is the original distance multiplied by the scale factor.

If the scale factor is greater than 1, the term *stretch* makes sense. We still use the term *stretch* when the scale factor is less than 1, even though the graph appears to shrink. This is similar to saying that division by 3 has the same effect as multiplication by $\frac{1}{3}$.

Worked example 3.13

From the given graph of $y = f(x)$, construct the graph of:

a $y = f(x - 1)$ **b** $-y = f(x)$



Solution

Understand the problem

a y remains the subject of the new graph, so we need to look at how the x-coordinates would change to give the same y-coordinates.

b x remains unchanged, so we need to look at how the y-coordinates would change to give the same x-coordinates.

Make a plan

The graph of $y = f(x)$ passes through $(0, 1)$, so that $1 = f(0)$

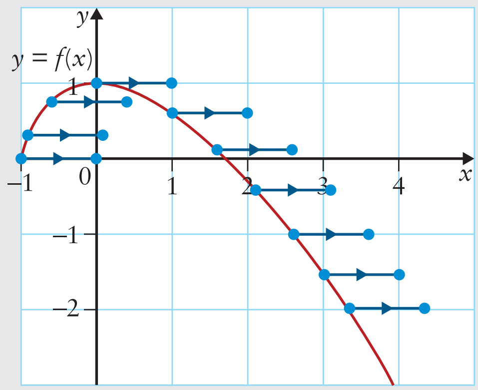
a Considering $y = f(x - 1)$, what value of x would produce the same result of $y = 1$? We will use this to understand the more general pattern of changes. Then draw guides to represent the change.

b Considering $-y = f(x)$, what value of y would be obtained from $x = 0$?

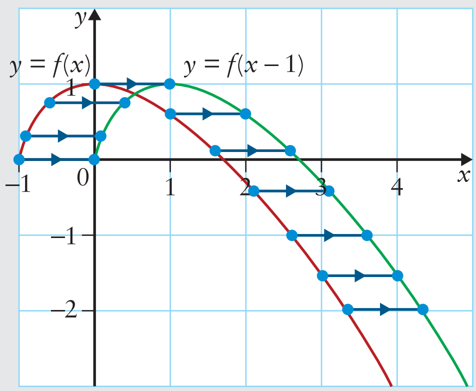
Carry out the plan

a $1 = f(0)$ and $y = f(x - 1)$, so choosing the value $x = 1$ would provide $x - 1 = 0$ as required.

We expect the point $(0, 1)$ on the original graph to correspond to $(1, 1)$ on the new graph. In general, x-coordinates would move 1 to the right as shown.

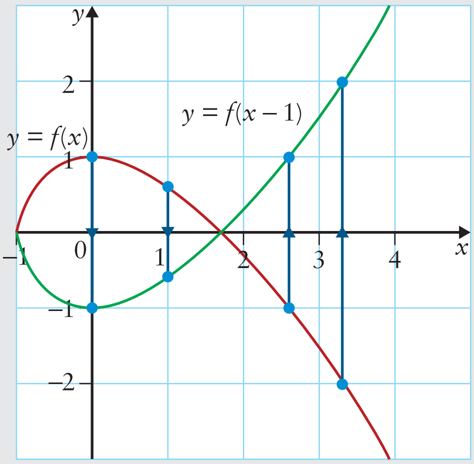


We can use this construction to trace the graph of $y = f(x - 1)$



b $1 = f(0)$ and $-y = f(x)$, so the value $y = -1$ would follow from $x = 0$ as required.

We expect the point (0, 1) on the original graph to correspond to (0, −1) on the new graph. In general, y-coordinates would change sign compared with the original graph, producing a reflection in the x-axis.



Look back

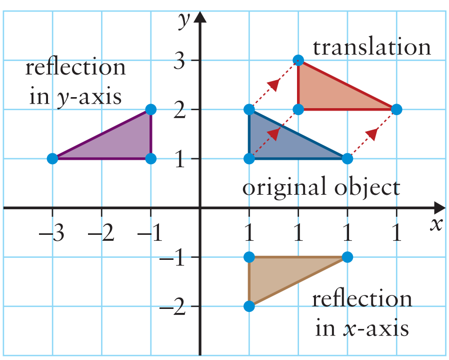
We can confirm our answers, again using the idea of compensation.

a Subtracting 1 from x (creating $x - 1$) while keeping the original value requires x itself to be increased by 1.

b Taking the negative of a negative reverts to the original value.

A transformation that moves all points a fixed distance in one direction is called a **translation**.

A transformation that swaps a y-coordinate for its negative is called a **reflection in the x-axis**. One that swaps an x-coordinate for its negative is called a **reflection in the y-axis**.

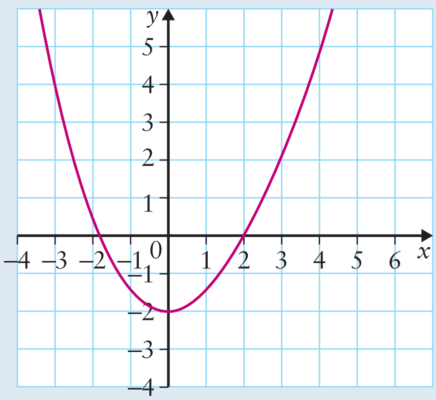


Practice questions 3.2.6

1 Make two accurate copies of this graph, which is of $y = f(x)$

To the first copy, add the graphs given in parts a and b.

To the second copy, add the graphs given in parts c, d and e.



a $y = f(x + 1)$ **b** $-y = f(x)$

c $y - 1 = f(x)$ **d** $y = f\left(\frac{1}{2}x\right)$

e $2y = f(x)$

Thinking skills

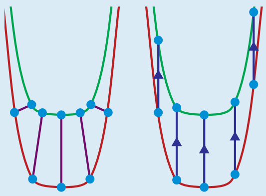
Fact Q2

When a graph has rotational or line symmetry it is possible for a particular transformation to be achieved in more than one way.

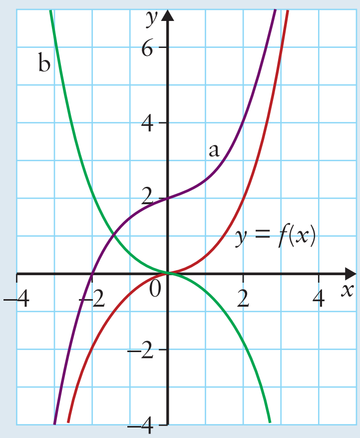
Hint Q2

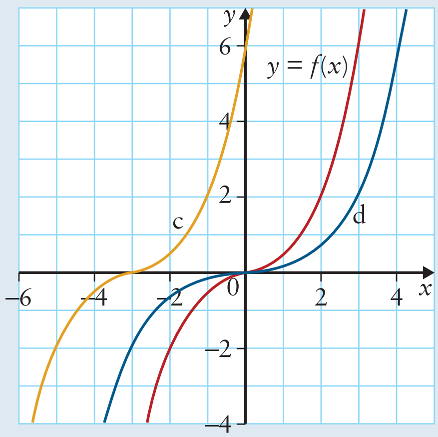
Vertical translations can be difficult to identify because the eye is drawn to the curves apparently getting closer together. Draw vertical lines between corresponding points to clearly identify the behaviour.

For example, compare the shortest distances on the left, which are more likely to catch the eye than the vertical distances shown on the right. Those on the right are all the same length.



2 Find transformed functions to describe each of the graphs a, b, c and d in terms of the graph of $y = f(x)$. Where appropriate, give two possible solutions.





3 Without drawing graphs, find the functions that perform each of the following transformations on $f(x)$.

a A translation of 3 units upwards

b A translation of 5 units to the left

c A stretch of scale factor 4 in the direction of the x-axis, with the y-axis as the line of invariant points

d A stretch of scale factor $\frac{1}{4}$ in the direction of the y axis, with the x-axis as the line of invariant points

4 $y = f(x)$ describes a straight line given by $f(x) = x + 1$. Which of the following describe identical lines?

A $y = f(x - 2)$

B $y + 2 = f(x)$

C The original graph is translated 2 units to the right to produce the new graph.

D The original graph is translated 2 units down to produce the new graph.

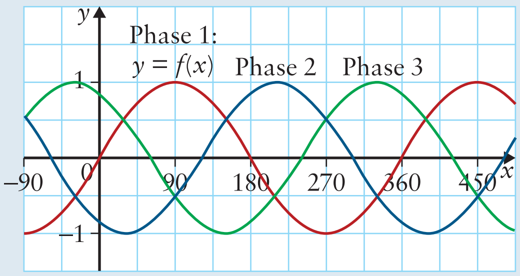
5 The graph shows the variation in (scaled) voltage for three-phase electricity based on the angle, x, of rotation of a generator. Phase 1 has the equation $y = f(x)$

Identify which of Phase 1, Phase 2 or Phase 3, if any, are described by the equations:

a $y = f(x + 120)$ b $y = f(x + 180)$ c $y = f(x + 240)$

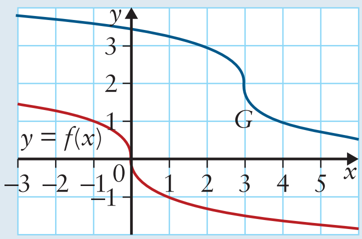
d $y = f(x - 120)$ e $y = f(x + 360)$ f $-y = f(x + 180)$

Challenge Q5f



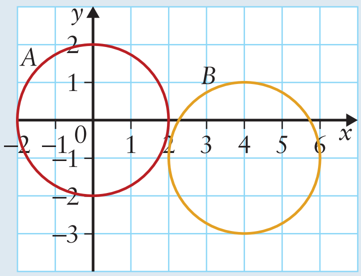
6 What transformed function would describe the graph G in terms of the original $y = f(x)$?

Challenge Q6



Challenge Q7

7 The methods used for transformations can be applied to relations that are not functions. The circle A has equation $x^2 + y^2 = 4$. Suggest an equation for circle B, which is the result of a translation of 4 units to the right and 1 unit down.



Self assessment

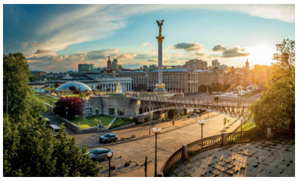
- I understand that a relation is a connection between an element in one set and an element in another set.
- I know that the initial set is called the domain and the target set is the range.
- I know that a relation is often described in terms of a rule, but may be defined by the connections themselves.
- I can illustrate a relation with a grid, a mapping diagram, a set of ordered pairs and a Cartesian graph.
- I understand that in the context of relations 'many' means 'more than one'.
- I can classify relations as one-to-one, many-to-one, one-to-many or many-to-many.
- I can recognise, use and apply one-to-one relations.
- I know that functions make connections that provide a unique output for any input within the domain.
- I know that functions may be defined as one-to-one or many-to-one relations.
- I know that inverse functions can be found for one-to-one functions.
- I can use $f(x)$ notation for functions and $f^{-1}(x)$ notation for inverse functions and recognise the efficiency of the notation in certain situations.
- I can identify functions from their graphs, but also recognise the limits of the method.
- I can find the equivalence between transformed functions and their graphs and describe corresponding stretches, translations and reflections in the axes.

Check your knowledge questions

1 Use a mapping diagram to connect the largest cities in Europe to their countries:

Moscow, London, Saint Petersburg, Berlin, Madrid, Kyiv, Rome, Paris, Bucharest, Minsk, Vienna, Hamburg, Warsaw, Budapest, Barcelona, Munich, Kharkiv, Milan

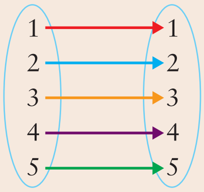
Austria, Belarus, France, Germany, Hungary, Italy, Poland, Romania, Russia, Spain, Ukraine, United Kingdom



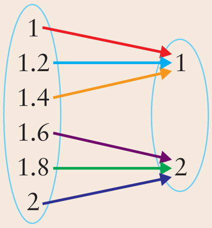
Kyiv

2 Create a set of ordered pairs for each relation shown in these mapping diagrams.

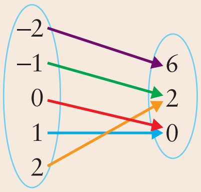
a



b



c



3 In each case, decide whether the relation is one-to-one, many-to-one, one-to-many or many-to-many.

a Students in a class mapped to their birthdays

b Golfers in a championship mapped to their total scores for four rounds

c Winners of The French Open men's singles mapped to the year in which they won

d The perimeter of a square mapped to its area

e The perimeter of a rectangle mapped to its area

4 Represent each of the following sets of ordered pairs on a Cartesian graph and state whether the relation is one-to-one, many-to-one, one-to-many or many-to-many.

a $\{(-3, 4.5), (-2, 2), (-1, 0.5), (0, 0), (1, 0.5), (2, 2), (3, 4.5)\}$

b $\{(0, 10), (1, 8), (2, 6), (3, 4), (4, 2), (5, 0)\}$

c $\{(0, 0), (1, 0), (1, 1), (0, 1)\}$

d $\{(-3, -9), (-2, -3), (-1, 0), (0, 0), (1, 0), (2, 3), (3, 9)\}$

5 Which of the following relations are functions when applied to the domain of positive integers?

a is a multiple of

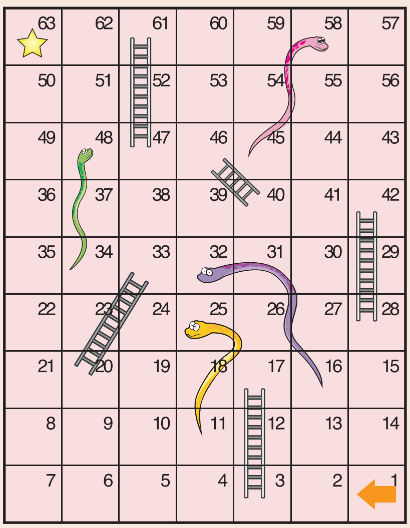
b is four more than

c is a factor of

d is the square of

e is the double of

6 In a game of Snakes and Ladders, you start with your counter on square 1, roll a die and advance the number of squares shown. You follow the sequence of numbers unless your counter lands at the foot of a ladder or on the head of a snake. If you land at the foot of a ladder then you climb immediately to the top of it. If you land on the head of a snake, you drop immediately to the end of the tail.

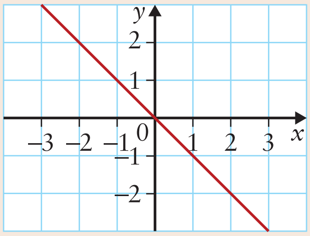


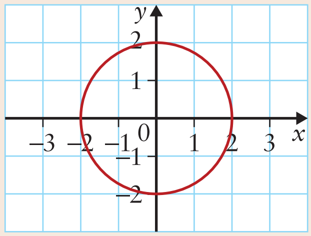
a What is the smallest possible total of scores on the die with which you could reach the star on the board shown on the previous page?

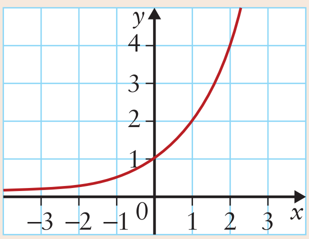
b Is the number of the square your counter is on a function of the total of scores of the die up to that point?

c Would your answer to part b be different if there were no snakes and no ladders?

7 For each of the following graphs, decide whether or not y is a function of x. If it is a function, state whether it has an inverse function for the full graph as shown.

a


b


c


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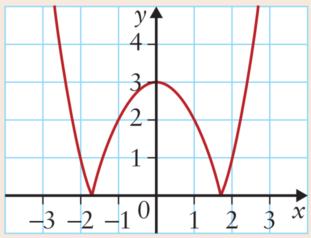

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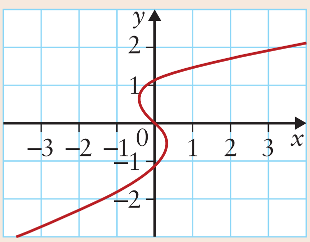

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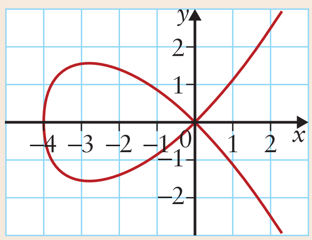
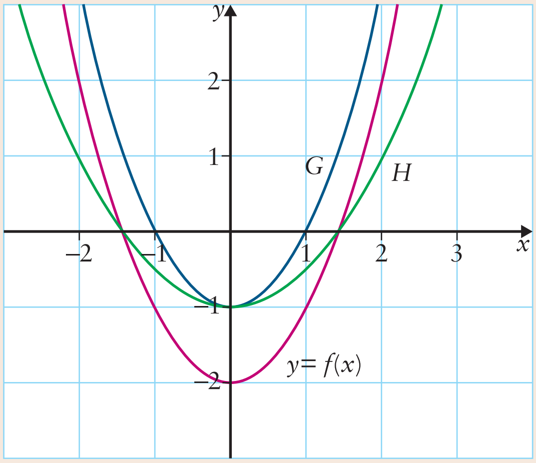


8 Describe graphs G and H in terms of $y = f(x)$



9 Write down the description of the transformations given by:

a $y = f(x)$ becoming $5y = f(x)$

b $y = f(x)$ becoming $y = f(x - 3)$

c $y = f(x)$ becoming $y = f\left(\frac{x}{2}\right)$

d $y = f(x)$ becoming $y + 10 = f(x)$

e $y = f(x)$ becoming $y = f(-x)$