CHAPTER 7 ALGEBRA AND EQUATIONS

OVERVIEW

Students will cover the basic notation and terminology and will explore the meaning of algebraic expressions. They will evaluate and simplify algebraic expressions and recognise the meaning of algebraic expressions in context. Students will encounter simplifying expressions with grouping symbols and after this those with powers of the same variable. They will review and extend their work on evaluating expressions started in Chapter 6. Students will also work with indices. Finally they learn what solving an equation means and solve such equations in one- and two-steps considering algebraic equations and using algebraic equations to solve real-life problems. They will also be introduced to solving linear equations graphically

PREREQUISITES

Working with variables, knowledge of coordinate planes and material covered in Chapter 6.

TEXT QUESTIONS

1 What value of n makes this equation correct: $6(n + 2) = 24$

2 When I talk on my phone there is an initial fixed cost of 80 cents plus 40 cents for every minute I speak. If a call lasts m minutes and costs \$3.20, what is the value of m?

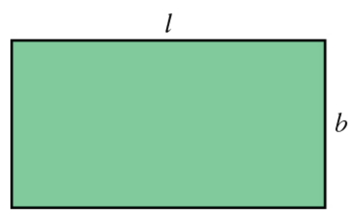
3 Simplify each expression.

a $6ab + 6a - 4b + ab - 5a$

b $5 + 2x + 10y - 2 - x + y$

4 Evaluate $5(2y + 1)$ given that $y = 3$

5 **a** Write an expression for the perimeter P of the rectangle in terms of its length l and width b.



b Find the perimeter when $l = 5$ and $b = 3$

Answers

1 2

2 6 minutes

3 **a** $7ab + a - 4b$ **b** $3 + x + 11y$

4 35

5 **a** $P = 2(l + b)$ **b** 16

GENERAL CHAPTER ADVICE

Suggested time and emphasis: 7–9 lessons. This depends on the nature of the group.

Points to stress

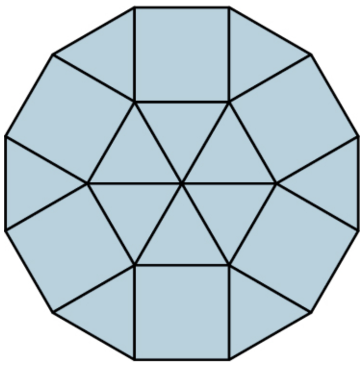
This is the second chapter introducing students to algebra concepts.

- Algebra makes working out problems more efficient by 'translating' sentences into equations and expressions.
- The difference between an expression and an equation: an equation is an expression that contains an equal sign (=). The equal sign indicates equivalence between the expressions or terms on each side of the sign.
- To find a solution to an equation (solving an equation) means finding that/those value(s) that will make the equation a true sentence rather than an open sentence.

EXTRA PRACTICE QUESTIONS

A Extra practice

1 **a** If the area of each square tile is s cm^2 and the area of each triangular tile is t cm^2, find the total area of the pattern.

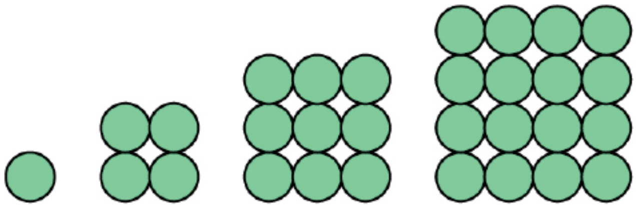


b If a pattern contained n squares and m triangles, what would its area be?

2 Expand and simplify

a $4(2n+1)+7$ **b** $5(10m+3)-20m+1$

3 **a** Complete the table of values for the pattern of circles below.



Number of circles on each edge (C)	1	2	3	4	5	6
Total number of circles (T)						

b Graph this set of ordered pairs on a coordinate plane.

c Find a rule for this pattern

4 Solve each equation.

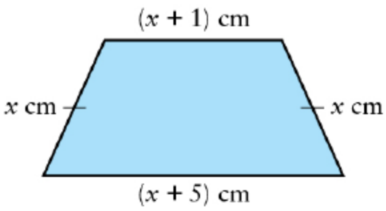
a $17 = 4x + 5$

b $2 = 24 - 2n$

c $3x - 2.1 = 3.0$

d $3(2x - 5) + 4x + 6 = 5x - 4$

5 Find the value of x if the perimeter of the trapezium is 42 cm.



Answers

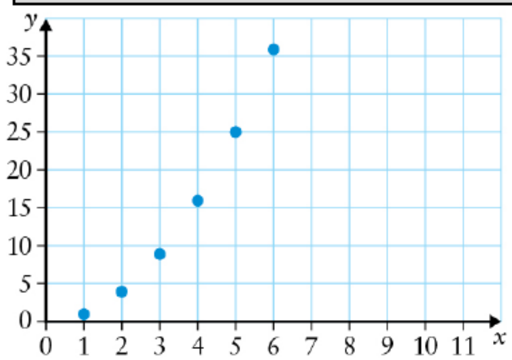
1 **a** $6s + 12t$ **b** $ns + mt$

2 **a** $8n + 11$ **b** $30m + 16$

3 **a**

Number of circles on each edge (C)	1	2	3	4	5	6
Total number of circles (T)	1	4	9	16	25	36

b



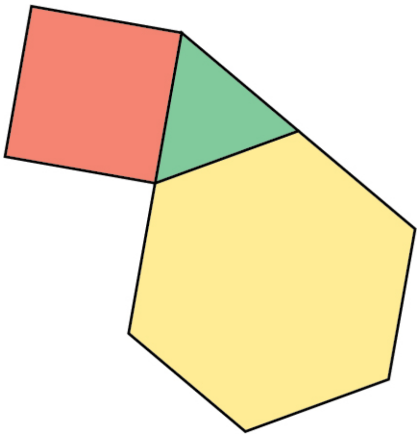
c $T = C \times C = C^2$

4 **a** $x = 3$ **b** $n = 11$ **c** $x = 1.7$ **d** $x = 1$

5 $x = 9$

B Challenge

1 The perimeter of the shape shown is 108 units.
The shapes are regular polygons.
What is the perimeter of the square?



2 A hire firm uses the following formula to work out the rental per day on one of their small vans.

$$\text{Rental per day} = €\left(25 + \frac{\text{kilometres travelled}}{2}\right)$$

a If you wish to hire the van for one day to travel to your destination and back a distance of 120 km, what will your rental cost be?

b If you have a budget for €110 only. What is the maximum distance you can drive?

3 Sophia plays a quiz game in which she scores 10 points for a correct answer but –5 points for an incorrect answer.

a If r is the number of right answers, and w is the number of wrong answers, her score S would be given by the rule:

A $S = 5r - 10w$

B $S = 10w - 5r$

C $S = 10r - 5w$

b What would Sophia's score be if she got:

i 3 right and 1 wrong

ii 5 right and 2 wrong

iii 5 right and 10 wrong?

c If Sophia answered 20 questions and her score was 20, how many questions did she answer correctly?

4 A concrete contractor uses wooden boards to 'box in' his concrete while it dries. In making a path, he records the amount of concrete he uses and the number of boards he lays down along the path. The results are given in the table.

Concrete (m^3)	No. of boards
2	7
3	12
4	17
5	22

a Find a rule connecting the amount of concrete with the number of boards used.

b A large project is estimated to need 32 m^3.
How much concrete would be boxed in by 32 boards?

Answers

1 48 units

2 **a** €85 **b** 170 km

3 **a** C

b **i** 25 **ii** 40 **iii** 0

c 8

4 **a** Number of boards = $5n - 3$ **b** $n = 7$ so 7 m^3 concrete needed

EXPLORES AND INVESTIGATIONS GUIDANCE

Remember that Explores are opportunities for students to 'discover' ideas and to become involved in the learning process. The suggested times can differ according to the groups in your class. ***A class discussion should follow every Explore.*** *Please remember that the approach is typical for all Explores:* ***Individual ↔ Group ↔ Class.*** *The individual explores, interacts with group, and groups interact with other groups in class.*

Explore 7.1

Duration: 5–10 minutes

Working out the repeated addition, we have:

x	1	2	3	4
y	4	8	12	16

A simpler way to describe the pattern is $y = 4x$.

For the second expression, replacing $x = 5$ results in:

$$y = \frac{(5+2)(5-2)}{5^2-4} = \frac{7 \cdot 3}{25-4} = 1$$

Repeating for $x = 6, 7, 8$ gives $y = 1$, since the whole fraction simplifies to 1.

The table of values is therefore:

x	5	6	7	8
y	1	1	1	1

and the rule is simply $y = 1$. You may want to mention that this pattern is called 'a constant' pattern.

During the discussion you may want to ask: 'What if $x = 2$?' to get a discussion regarding division by zero.

Explore 7.2

Duration: 5–10 minutes

The two possibilities here are:

$3 \times (4+5) = 27$ and $3 \times 4 + 5 = (3 \times 4) + 5 = 17$

Any other expression for which the order of operations matters, but is not clear or specified, would work. Examples are 'three plus four times six', which can evaluate to either 42 or 27, and 'sixteen divided by four plus four', which can evaluate to either 8 or 2, depending on which pair of numbers is put in brackets.

Expressions involving operations in which the order does not matter are also possible, as for instance 'three plus three plus three', which is unambiguously 9 (since addition is associative), or 'two times two times two'.

When reading expressions aloud, it can be useful to use pauses to suggest the correct order of operations, for example, 'three plus [*pause*] four-times-six' which suggests doing four times six first, then adding three to the result, that is, $3 + (4 \times 6) = 27$.

Brackets are used differently in everyday writing, for example: to give additional but not essential information, to give an aside, to give an explanation, etc.

Explore 7.3

Duration: 5–10 minutes

If two customers sit at a table, there will be two plates for each of them – so 4 plates – and one plate for the bread. This gives a total of 4 + 1 = 5 plates.

If the waiter collects 13 plates from a table and one of them is for the bread, then 13 – 1 = 12 plates are for the customers. Since there are two plates for each customer, this means that there were $\frac{12}{2} = 6$ customers at that table.

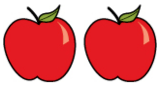
An algebraic approach can be followed: let p be the number of plates arranged for a table, and c be the number of customers at that table. The rule that relates p and c is: $p = 2c + 1$.
The first question can be answered by substituting $c = 2$ and evaluating: $p = 2 \times 2 + 1 = 5$
The second question can be answered by setting $p = 13$ and solving $13 = 2c + 1$ for c.

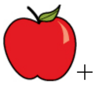
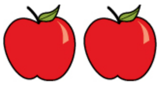
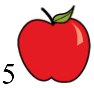
Explore 7.4

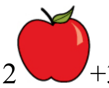
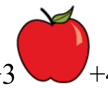
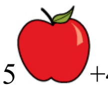
Duration: 5–10 minutes

Here, there are 5 apples in total.

In 'algebraic' notation:

 reads as 2 
and
 reads as 3 

When I consider the total:
2  + 3  =  + 
=  = 5 

For the question about apples and oranges, there are 9 fruits in total but the expression
2  +3  +4  can only partially be simplified to 5  +4 .

Explore 7.5

Duration: 5–10 minutes

At Al's Café, x sandwiches cost €2.50x, y drinks cost €1.50y, and z ice creams cost €2.00z.
In total, I would spend $A = 2.50x + 1.50y + 2.00z$ euros.
For the given numbers, the total cost would be $A = 2.50 \cdot 7 + 1.50 \cdot 12 + 2.00 \cdot 9 = €53.50$
At Jody's Deli, an expression for the total cost would be $J = 2.00x + 1.75y + 2.50z$. For the same purchase as before, the total cost is $J = 2.00 \cdot 7 + 1.75 \cdot 12 + 2.50 \cdot 9$ = €57.50, so Al's is the cheaper option.

Explore 7.6

Duration: 5–10 minutes

The total length of string required for the triangle is $L_{\text{triangle}} = l + l + 5 = 2l + 5$

For the trapezoid, it is $L_{\text{trapezoid}} = l + l + l + 15 = 3l + 15$.

The total length for both shapes would be $L_{\text{triangle}} + L_{\text{trapezoid}} = (2l + 5) + (3l + 15)$, but brackets can be removed as the total length is a sum:

$L_{\text{triangle}} + L_{\text{trapezoid}} = 2l + 5 + 3l + 15 = 5l + 20$, by collecting and adding like terms.

Explore 7.7

Duration: 5–10 minutes

a The length for each semicircle is $L = y + 3$, so for both semicircles the total length is:
$L_{\text{total}} = 2L = 2(y+3) = (y+3) + (y+3) = y + 3 + y + 3 = y + y + 3 + 3 = 2y + 6$

b The length for each triangle is $L = x + y + 3$, so for the three triangles the total length is:

$L_{\text{total}} = 3L = 3(x + y + 3) = 3x + 3y + 9$

It becomes apparent that each triangle requires $x + y$ metres of string *and* 3 metres of string.

Having spotted the pattern, for four triangles the length would be:
$L_{\text{total}} = 4L = 4(x + y + 3) = 4x + 4y + 12$

For 10 triangles:
$L_{\text{total}} = 10L = 10(x + y + 3) = 10x + 10y + 30$.

Generalising, the total length for m triangles, would be:

$L_{\text{total}} = mL = m(x + y + 3) = mx + my + 3m$

Students may simplify the expression slightly differently.

Explore 7.8

Duration: 5–10 minutes

The expression x^2 means $x \cdot x$, so there are two xs. x^5 means $x \cdot x \cdot x \cdot x \cdot x$ so there are five xs.

When working out the meaning of $x^2 \cdot x^5$, we replace each power with its expansion as repeated multiplication, $x^2 \cdot x^5 = (x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x)$. Like addition, multiplication is associative – for instance, $(2\cdot3)\cdot5=2\cdot(3\cdot5)$ – so the brackets are not necessary and can be removed. So $x^2\cdot x^5=(x\cdot x)\cdot(x\cdot x\cdot x\cdot x\cdot x)=x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x$.

In the latter expression, x is multiplied by itself $2+5=7$ times, so we can write this as single power of x as $x^2\cdot x^5=x^7$.

Applying the same process to the other parts of the exploration, we have:

a $x^3x^5=(\underbrace{x\cdot x\cdot x}_{3})\cdot(\underbrace{x\cdot x\cdot x\cdot x\cdot x}_{5})=\underbrace{x\cdot x\cdot x}_{3}\cdot\underbrace{x\cdot x\cdot x\cdot x\cdot x}_{5}=\underbrace{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x}_{8}=x^8$

b $a^2a^4=(a\cdot a)\cdot(a\cdot a\cdot a\cdot a)=a\cdot a\cdot a\cdot a\cdot a\cdot a=a^6$

c $y\times y^2=y\times(y\times y)=y\times y\times y=y^3$ since both symbols $\times$ and $\cdot$ stand for multiplication.

In the general case:

$$x^mx^n=(\underbrace{x\cdot x\cdot\cdots\cdot x}_{m})\cdot(\underbrace{x\cdot x\cdot\cdots\cdot x}_{n})=\underbrace{x\cdot x\cdot\cdots\cdot x}_{m}\cdot\underbrace{x\cdot x\cdot\cdots\cdot x}_{n}=\underbrace{x\cdot x\cdot\cdots\cdot x}_{m+n}=x^{m+n}$$

This is often called the First Index Law, and similar applications to division and powers yield the different index laws. For instance, we have:

$$(x^m)^n=\underbrace{(\underbrace{x\cdot x\cdot\cdots\cdot x}_{m})\cdot(\underbrace{x\cdot x\cdot\cdots\cdot x}_{m})\cdot\cdots\cdot(\underbrace{x\cdot x\cdot\cdots\cdot x}_{m})}_{n}=\underbrace{x\cdot x\cdot\cdots\cdot x}_{m\cdot n}=x^{mn}.$$

Students often confuse addition of indices with multiplication, which is worth discussion.

Explore 7.9

Duration: 5–10 minutes

Here, we have an unknown number of pencils in the bag and one pencil outside. Given that there are 5 pencils in total, there must be 4 pencils in the bag, because 4 is the number we must add to 1 to obtain 5.

In algebraic notation, we call x the unknown number of pencils in the bag.
So we can write $x+1=5$, and therefore $x=4$

The two questions have the same meaning although they are presented differently: one is given as a sentence while the other is an equation. Solving the equation $x+1=5$ actually means finding the number x that makes $x+1$ equal to 5.

Explore 7.10

Duration: 5–10 minutes

In this question, if there are 10 pencils in total and the two bags each contain the same number of pencils, there must be 5 pencils in each bag.

In algebraic notation, we call x the unknown number of pencils in each bag, and we can therefore say $x + x = 10$

The two questions have the same meaning although presented differently (as a sentence and as an equation). $x + x = 10$ can be written as $2x = 10$, so the unknown number of pencils in each bag is the number that makes $2x$ equal to 10.

Explore 7.11

Duration: 5–10 minutes

From our knowledge of motion, we can relate the distance d travelled by a bird that moves at constant speed v for a time t using the formula $d = v \cdot t$

In this example, the speed and the distance travelled are known, and the question requires that we find the time, so the formula has to be rearranged. This is called 'solving for t'.

It is preferable to rearrange the general formula to make the desired variable the subject, that is $t = \frac{d}{v}$, and then substitute given values to solve for t: $t = \frac{6000}{50} = 120$ hours.

But it is also possible to replace the known values first and then solve. Doing this gives $6000 = 50 \cdot t$, and the question becomes 'what number, multiplied by 50, gives 6000?' The answer is 120 hours.

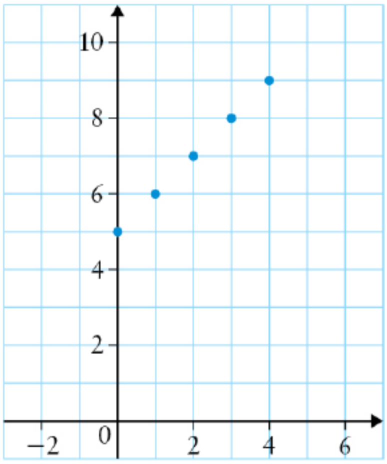
Explore 7.12

Duration: 5–10 minutes

The table of values for $y = x + 5$ is:

x	0	1	2	3	4
y	5	6	7	8	9

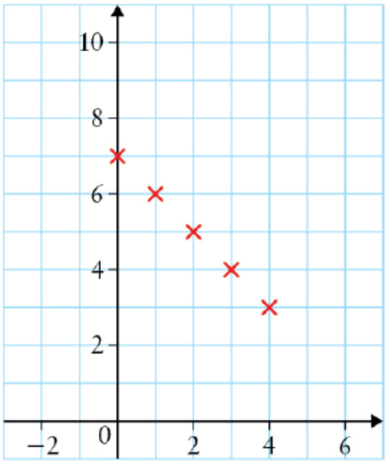
Plotting these five points gives



Repeating the same process for $y = 7 - x$, gives:

x	0	1	2	3	4
y	7	6	5	4	3

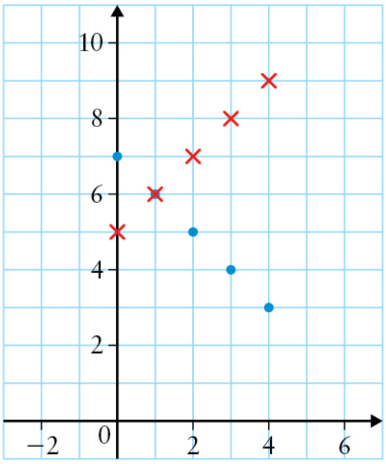
The plotted points are:



By looking at the two graphs, we observe that the point (1, 6) belongs to both graphs. In other words, when $x = 1$ both $y = x + 5$ and $y = 7 - x$ evaluate to $y = 6$.

This tells us that the solution to the equation $x + 5 = 7 - x$ is $x = 1$

This is especially clear if $y = x + 5$ and $y = 7 - x$ are plotted on the same set of axes:



In fact, we can see that the point (1, 6) is marked both with a dot and with a cross.

For detailed descriptions of criteria listed in the following Investigations, check the MYP Mathematics Guide 2020, pages 34–37

Investigation 7.1

Possible outcomes of this investigation depend on students' backgrounds. For instance, in the Italian language the suggested apostrophe symbol ' is used to truncate an article before a vowel, but only for feminine nouns ('una amica' becomes 'un'amica', but 'un amico' has no apostrophe) or the last syllable in a word ('poco' becomes 'po').

This investigation can be a way to introduce students to conventions used in mathematics. For example, $2 \times y$ can also be written as $2 \cdot y$ or simply $2y$.

Take the word 'Expand' in the expression $3(x+2y)$ in this chapter. Obviously, it does not mean make its size larger!

Students can also remember their long division and the different ways the process is organised. This promises to be a rich sharing activity when students bring their views to their groups and to class.

Can be assessed using Criteria A, B, C, D

Investigation 7.2

Some ambiguous expressions could be $x \div 2y$ or $5 \div 3z$

They both involve division, and there is – apparently – a product after the $\div$ symbol in both.
The way to remove the ambiguity is to use brackets or, preferably, to use fraction bars:

$$x \div (2y) = \frac{x}{2y} \text{ whereas } (x \div 2)y = \frac{x}{2}y \text{ or } \frac{xy}{2}$$

$5 \div (3z) = \dfrac{5}{3z}$ whereas $(5 \div 3)z = \dfrac{5}{3}z$ or $\dfrac{5z}{3}$

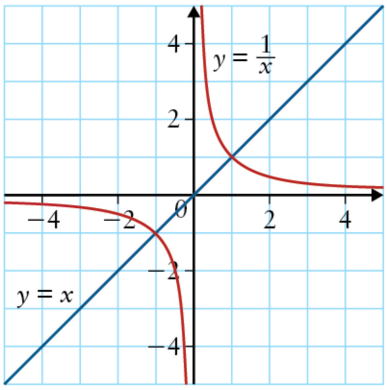
The investigation leads to the importance of using grouping symbols to insure clarity of expressions. This is especially important when students use calculators because they do not 'see' what convention the calculator programmers used.

Can be assessed using Criteria A, B, C, D

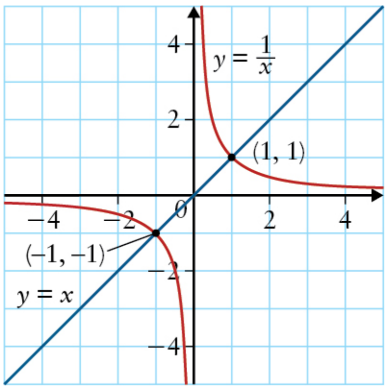
Investigation 7.3

Students will probably have at least a GDC. Other software such as GeoGebra or others are also commonly available not to mention many internet apps.

Graphing separately the left and right sides of the first equation, we have:

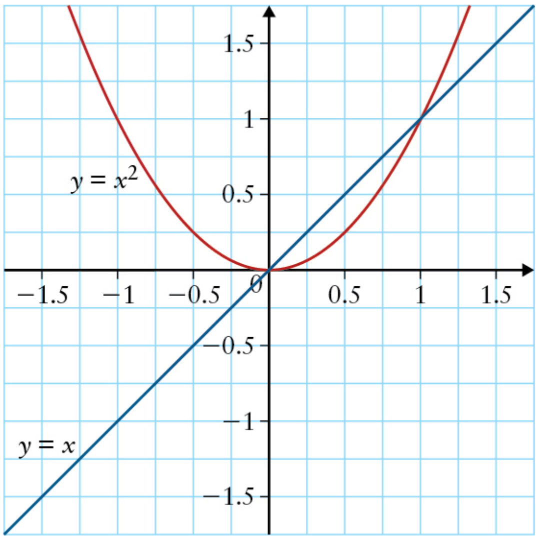


We can see from the graph that there are two intersection points, with coordinates (–1 , –1) and (1, 1).

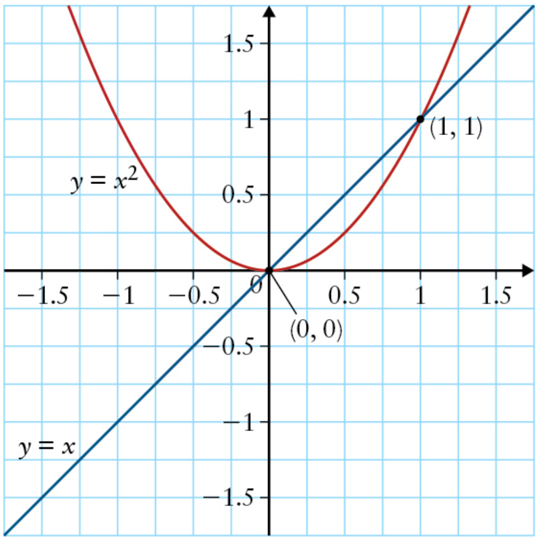


The solutions of the equation $\dfrac{1}{x} = x$ are therefore $x = -1$ and $x = 1$, i.e., the ***x*-coordinates** of the intersection points. In justifying the graphical solution, it is expected that students refer to the idea of the solution of an equation – It is the value that satisfies both sides of an equation to make a true sentence.

For the second equation, it is a similar approach. Graphing separately the left and right sides of the second equation, we have



We can see from the graph that there are two intersection points, with coordinates (0, 0) and (1, 1).



The solutions of the equation $x^2 = x$ are therefore $x = 0$ and $x = 1$, i.e. the ***x*-coordinates** of the intersection points.

Can be assessed using Criteria A, B, C, D

EXTRA GROUP ACTIVITIES

Activity 1

It is recommended that this activity be done as in-class group activity where different groups check other groups' work.

Here is an example of a process:

Begin with x, multiply it by 8, add 4, subtract 8, divide the result by 4 and find the value of this expression if x is equal to 7.

Here is a correct record of the calculations involved:

$$\boxed{x} \xrightarrow{\times 8} \boxed{8x} \xrightarrow{+4} \boxed{8x + 4} \xrightarrow{-8} \boxed{8x - 4} \xrightarrow{\div 4} \boxed{2x - 1} \xrightarrow{x = 7} \boxed{2(7) - 1 = 13}$$

Apply the method shown above to different processes:

1 Begin with x, multiply by 6, add 2, subtract 6, divide by 2, let x equal 8.

2 Begin with y, multiply by 10, add 5, multiply by 2, subtract 20, divide by 10, let y equal 5.

3 Begin with $3d$, add $5d$, subtract 7, multiply by 3, subtract 24, let d be 4.

4 Begin with $10k$, subtract k, multiply by $2k$, add 18, divide by 2, let k equal -3.

The answers are:

1 x, $6x$, $6x + 2$, $6x - 4$, $3x - 2$; when $x = 8$, answer is 22.

2 y, $10y$, $10y + 5$, $20y + 10$, $20y - 10$, $2y - 1$; when $y = 5$, the answer is 9.

3 $3d$, $8d$, $8d - 7$, $24d - 21$, $24d - 45$; when $d = 4$, the answer is 51.

4 $10k$, $9k$, $18kk$ (or $18k^2$), $18kk + 18$ (or $18k^2 + 18$), $9kk + 9$ (or $9k^2 + 9$); when $k = -3$, the answer is 90.

Activity 2: Maths apps and games

The website Quia has a Concentration activity for solving simple equations.

Download the activity. This is a matching/memory game. Click on two blank cells in the grid. If they show an equation and a correct solution, they are paired. If not, they turn over again. The aim is to pair up matching equations and solutions until all are turned over, so you need to remember where unmatched equations and solutions are.