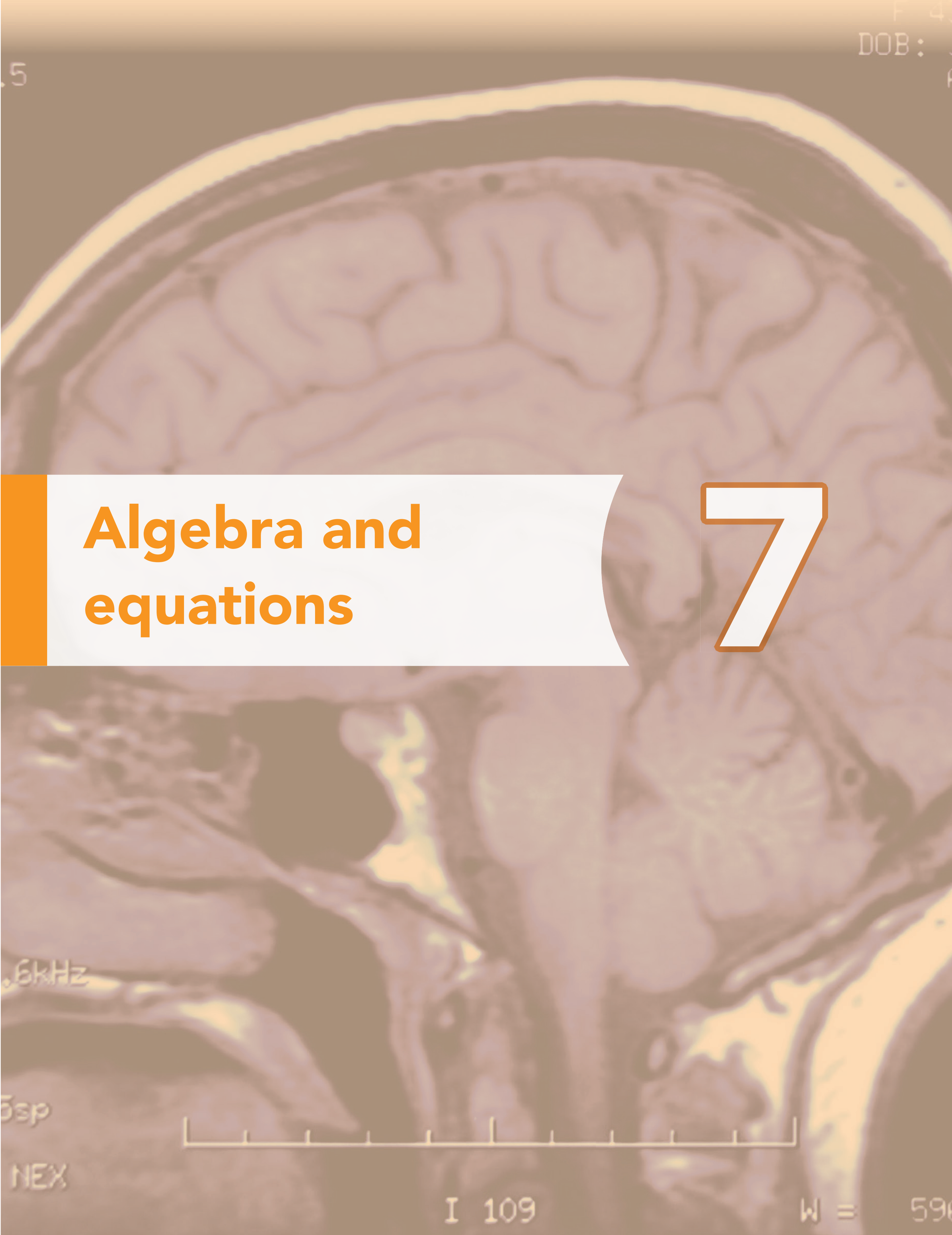


7 Algebra and equations

7 Algebra and equations

KEY CONCEPT

Form

RELATED CONCEPTS

Equivalence, Representation, Simplification

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Generalising patterns and representing them in a simplified form helps us to find innovative solutions to real-life problems.

Inquiry questions

Conceptual

- Can mathematics be regarded as a language?

Factual

- How do mathematical skills support technical advancement?

Debateable

- Can all problems be tackled efficiently with mathematical tools?

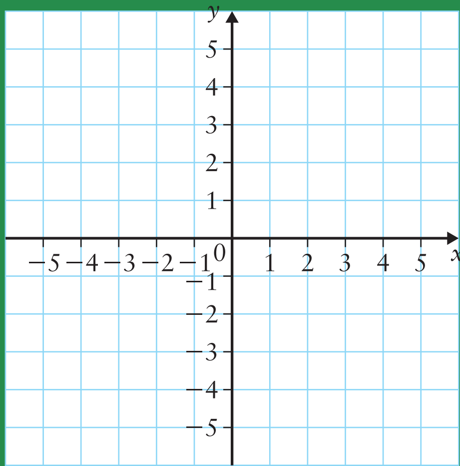
Do you recall?

1 What is a pattern? Name three ways of describing a pattern.

2 What advantages do algebraic rules have over the other methods to describe a pattern?

3 a How are pairs of numbers placed on the Cartesian plane?

b Copy the set of axes below, and plot the points (−1,2), (3,3), (−2,−2) and (0.5, −1.5).



4 a How do we graph a rule?

b Graph the rule $y = 2x + 2$

7.1 Algebraic notation

Fact

When numbers and variables are combined using the four operations $+, -, \times, \div$ and powers, we obtain an algebraic expression.

Explore 7.1

Look at the expression $y = x + x + x + x$

Copy and complete the table of values for this expression.

x	1	2	3	4
y				

By looking at the table of values, can we write down another rule for this pattern? Which of the two algebraic rules is simpler?

This expression is more complicated: $y = \dfrac{(x + 2) \times (x - 2)}{x \times x - 4}$

Copy and complete the table of values for this expression.

x	5	6	7	8
y				

By looking at the table of values, can you write down another rule for this pattern? Which of the two algebraic rules is simpler?

A labyrinth looks like a complicated structure, but there is only one path to follow. If you keep walking, you will get to the centre. Sometimes, algebraic expressions also look complicated, but if you work through them step by step, you can make them simpler.

In Explore 7.1, we saw that algebraic expressions sometimes look more complicated than they really are. We found a way to make them easier to work with.

In this chapter, we will learn how to make an algebraic expression simpler without using a table of values.

7.1.1 Terminology

A variable is a letter or symbol used to represent a quantity whose value can vary.

In an expression that is a sum, such as $x + 34 + b$, the quantities that are being added are called **terms**. The terms of the expression $x + 34 + b$ are x, 34 and b.

In an expression that is a product, such as $2 \times a \times n$, the quantities that are being multiplied are called **factors**. The factors of the expression $2 \times a \times n$ are 2, a and n.

Connections

You learned about using variables to represent numbers in Chapter 6.

Reminder

The words term and factor are not synonyms! Use them accurately.

7.1.2 Abbreviations

When we write multiplications in algebra, we leave out the × sign when possible:

- in the product of a number and a variable: $12 \times n = 12n$
- in the product of two variables: $a \times b = ab$
- with brackets: $3 \times (x + 1) = 3(x + 1)$ and $a \times (b + c) = a(b + c)$

When we multiply two numbers we need to keep the multiplication sign:

- 3×7 cannot be written as 37

Often the dot symbol · is used for multiplication:

- $3 \cdot 7 = 3 \times 7$

When we multiply a variable by the number 1 or −1, we do not write the number 1:

- $1 \times n = n$
- $-1 \times n = -n$

When we multiply a variable by the number 0, we do not write the variable:

- $0 \times n = 0$

For division, we usually use the fraction sign instead of the ÷ sign:

- $3 \div y = \frac{3}{y}$
- $a \div b = \frac{a}{b}$

If we divide a variable by a number, we have two ways of writing the same expression:

- $x \div 3 = \frac{x}{3}$
- $x \div 3 = \frac{1}{3}x$

When we divide a variable by the number 1, the fraction is unnecessary:

- $\frac{n}{1} = n$

For repeated multiplication, or powers, we use the same notation as with numbers:

- $x \times x = x^2$
- $x \times x \times x = x^3$

In the power x^n, x is called the **base** and n is called the **exponent** or **index**. The plural of index is **indices**.

Fact

In the product of a number and a variable, we always write the number first.

Both $12 \times n$ and $n \times 12$ are written as $12n$.

Hint

Distinguish between $3 \cdot 7$ and the decimal 3.7.

Fact

Out loud, $a \div b = \frac{a}{b}$ is often read as 'a over b'.

Fact

Out loud, x^2 is read as 'x squared' and x^3 is read as 'x cubed'.

Communication skills

Explore 7.2

If you hear someone say ‘three times four plus five’, how would you write this down? Is this the only way?

Can you come up with other expressions that would sound the same when read out loud but could be written in different ways?

When we read expressions with brackets out loud, we must make sure we are very clear. One way to read $a(b + c)$ is ‘a, open bracket, b plus c, close bracket’. Can you think of any other ways that make the expression clear?

Why do we use brackets? How are brackets used in everyday writing? Is this different to how they are used in mathematics?

Worked example 7.1

Rewrite each expression without using multiplication or division signs.

a $7 \times b$ **b** $3 \cdot x$ **c** $3 \div a$ **d** $(3 + x) \div y$

Solution

a To write an algebraic product without using a multiplication sign, we leave out the sign and write the factors next to each other.

$7 \times b = 7b$

b $3 \cdot x = 3x$

c To write a division without using a division sign, we write it as a fraction. The number or variable to the left of the division sign is the numerator and the number or variable to the right of the division sign is the denominator.

$3 \div a = \dfrac{3}{a}$

d If there is an expression in brackets, the entire expression becomes the numerator or denominator.

$(3 + x) \div y = \dfrac{3 + x}{y}$

Learning vocabulary and grammar is vital to be able to communicate effectively, both with spoken languages and the language of mathematics.

Thinking skills

Communication skills

Investigation 7.1

In any language, there are conventions that we must be familiar with. Can you think of some of these conventions:

- in your native language (if it is not English)
- in the English language
- in the language of mathematics?

You could start with the apostrophe symbol ’. In how many different ways have you seen this symbol used?

7.1.3 Grouping symbols

In Chapter 1, you learned about the order of operations and about using brackets when you need to override the normal order of operations.

There are other grouping symbols you can use to make your expressions clear. Two that you are likely to see are:

- the fraction bar —
- the square root sign $\sqrt{}$

The expressions $\frac{4+7}{3}$ and $\frac{1}{x+1}$ are interpreted as $\frac{(4+7)}{3}$ and $\frac{1}{(x+1)}$

The expression $\sqrt{4+9}$ means $\sqrt{(4+9)}$

Reminder

The order of operations is: brackets, powers, then multiplications and divisions, then additions and subtractions.

Reflect

'Algebraic notation is more precise than any language.'

Do you agree with this statement?

Thinking skills

Communication skills

Investigation 7.2

Look at the expression $3 \div 2x$

Some people would say that because the 2 and the x have no symbol between them, they should be treated as a single entity, so the expression could be written as $\frac{3}{2x}$

Other people would say that because division and multiplication have the same priority, they should be carried out from left to right, so the expression could be written as $\frac{3}{2} \times x$ or $x\left(\frac{3}{2}\right)$

Write some expressions that could be interpreted in more than one way. How could you rewrite your expressions so that the meaning is clear?

Reminder

The rules you have learned for numbers apply in exactly the same way to expressions containing variables. Since variables stand for numbers, they obey the same rules as numbers.

Practice questions 7.1

1 Write each expression without using a multiplication sign.

a $3 \times g$	**b** $n \times 2$	**c** $3 \cdot y$	**d** $14 \cdot b$
e $x \cdot (-1)$	**f** $-3 \times g$	**g** $1 \cdot y$	**h** $2 \cdot (y+z)$
i $\frac{1}{2} \times w$	**j** $3 \times a + 4 \times b$	**k** $4 \cdot a - 5 \cdot b$	**l** $1 \times a + 10$

2 Write each expression as a fraction.

a $3 \div g$	**b** $n \div 2$	**c** $x \div 3$	**d** $15 \div b$
e $2b \div 3$	**f** $x \div q$	**g** $x \div 1$	**h** $1 \div x$
i $2 \div (y+z)$	**j** $(x+1) \div (x-1)$		**k** $3 \div (2x)$

Fact Q2g and Q2h

Division is not commutative. This means that the result of a division changes if the order of the numbers is reversed. For example, $6 \div 3 = 2$ but $3 \div 6 = 0.5$

Connections

In calculators, the order of operations is built in by the manufacturer.

Try evaluating $\frac{12}{2 \times 1.5}$ by entering $12 \div 2 \times 1.5$

What result do you get? What is the correct result? If your calculator gave a different result, why do you think that happened?

The calculator interprets $12 \div 2 \times 1.5$ as $(12 \div 2) \times 1.5$, but $\frac{12}{2 \times 1.5}$ means $12 \div (2 \times 1.5)$

When in doubt, always use brackets. Most calculators also have a fraction template.

3 Starting from the numbers 2 and 3, we can make larger numbers by using different combinations of operations and brackets. For instance, we could make:

$2 + 3 = 5$

$2 \times 3 = 6$

$2^3 = 2 \times 2 \times 2 = 8$

$3^2 = 3 \times 3 = 9$

The largest possible number is 9.

Find the largest number that can be made with each group of numbers.

a 4 and 5 **b** 1 and 3 **c** 2, 3 and 4 **d** 1, 3 and 5.

4 Rewrite each expression, showing all multiplication and division signs and all brackets. For example, $3x + 5 = 3 \times x + 5$, $\frac{1 + x}{3} = (1 + x) \div 3$

a $2s + t$ **b** $2(s + t)$ **c** $3(x - y)$ **d** $3x - y$

e $3xy$ **f** $\frac{3x}{y}$ **g** $2x - 4y$ **h** $2(x - 4y)$

i $2 + \frac{x}{3}$ **j** $\frac{2 + x}{3}$ **k** $x - \frac{4 + s}{w}$ **l** $x - 4 + \frac{s}{w}$

m $x^2 + 1$ **n** $(x + 1)^2$ **o** $a^2 - b^2$ **p** $(a + b)(a - b)$

q $\frac{q + b}{q - b}$ **r** $\frac{q + b}{q} - b$ **s** $\frac{1}{2s}$ **t** $\frac{1}{2}s$

u $\frac{s}{2}$ **v** $\frac{3(x + 1)}{y}$ **w** $\frac{3x}{2y}$ **x** $\frac{3x}{2}y$

y $\frac{3}{2}xy$ **z** $\frac{3}{2xy}$

5 Explain the meaning of each expression in question 4, using ‘first’ and ‘then’ to clarify the order of operations. For example, $3x + 5$ means ‘first multiply 3 by x then add 5’ and $\frac{1 + x}{5}$ means ‘first add 1 and x then divide the result by 5’.

7.2 Algebraic expressions

In this section, we will learn how to evaluate, simplify and expand algebraic expressions.

7.2.1 The meaning of algebra

Explore 7.3

When tables are set at a restaurant, two plates are arranged for each customer. There is also one plate of bread for each table.

If two customers sit at a table, how many plates will be arranged on their table? Explain how you know.

After dinner, the waiter collects 13 plates from another table. How many customers were seated at that table? Explain how you know.

Connections

Remember Pólya's four-step problem-solving process from Chapter 2 as you tackle this question: understand the problem, make a plan, carry out the plan, look back.

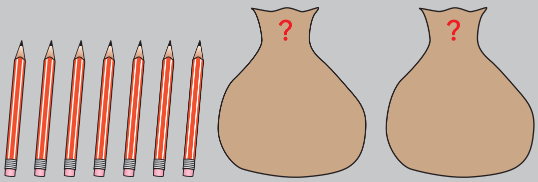
When you worked through the problems in Explore 7.3, you might have thought of using algebra as part of your plan. Algebra can help us answer questions like this in an efficient way.

Worked example 7.2

A box of pencils has been delivered to a classroom.

The pencils have been packaged in two different ways:

- There are seven loose pencils.
- There are two identical bags, each containing the same unknown number of pencils.



The label on the box says, 'weight of one pencil = 10 grams, total shipping weight = 1570 grams'.

Write an equation to represent this information.

Solution

To write an equation to represent the information, we need a way to represent the number of pencils in each bag. This number could be anything, say 12, 57 or 132. We can represent this number with a variable, x.

x = number of pencils in each bag

The information we know is:

- We have 7 loose pencils.

Reminder

We could use any letter to represent the number of pencils, for example, p for pencil or u for unknown. x is a common letter to use.

- We have 2 bags, each containing x pencils.
- One pencil weighs 10 grams.
- All the pencils together weigh 1570 grams.

All the pencils are identical, so we can relate the number of pencils to their weight. If one pencil weighs 10 grams, then two pencils will weigh $2 \times 10 = 20$ grams, and x pencils will weigh $x \times 10 = 10x$ grams. So:

Weight of pencils in each bag $= 10x$

There are two bags, so:

Weight of pencils in both bags $= 10x + 10x$

There are also seven loose pencils, so:

Weight of loose pencils $= 7 \times 10$

The total weight of all the pencils is the sum of the weights of the pencils in the two bags and of the loose pencils:

Total weight of pencils $= 10x + 10x + 7 \times 10$

We now have an algebraic expression for the total weight of pencils in the box. There is one piece of information that we have not used yet: the total weight of all the pencils is 1570 grams. We can equate this expression to the known weight to make an **equation**.

$$10x + 10x + 7 \times 10 = 1570$$

Reminder

When we build algebraic expressions we often drop the units.

Fact

Later in the chapter, we will **solve** the equation. This means finding the value of x that makes $10x + 10x + 7 \times 10$ evaluate to 1570.

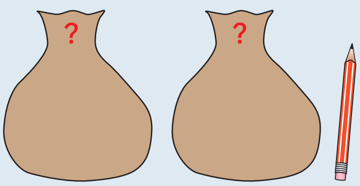
Reflect

What are the advantages of using x to represent a unknown quantity?

Practice questions 7.2.1

1 Write an expression for the total number of pencils using variables.

a



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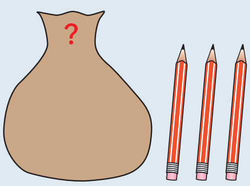

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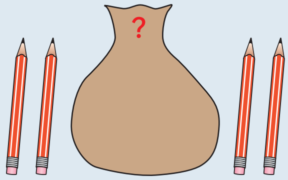

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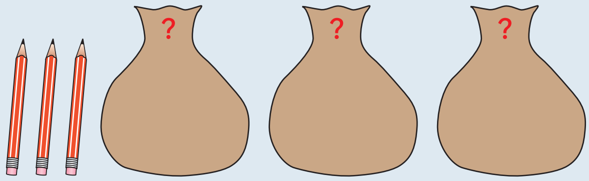
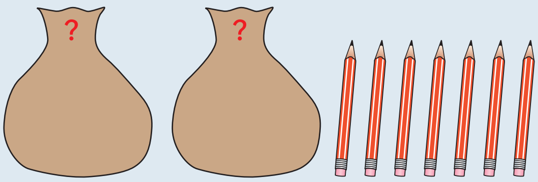
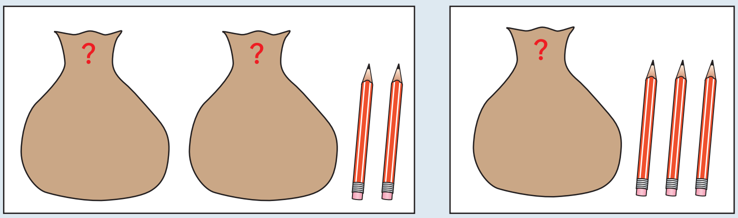


e

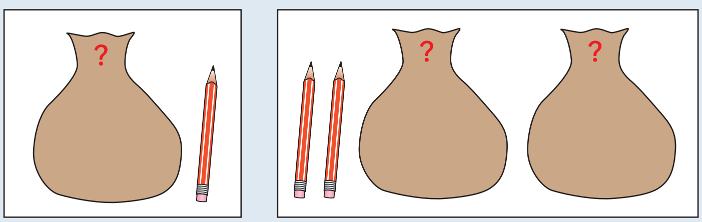


2 For each set of two boxes, write an expression for the total number of pencils using variables and brackets. For example:



$(x + x + 2) + (x + 3)$

a



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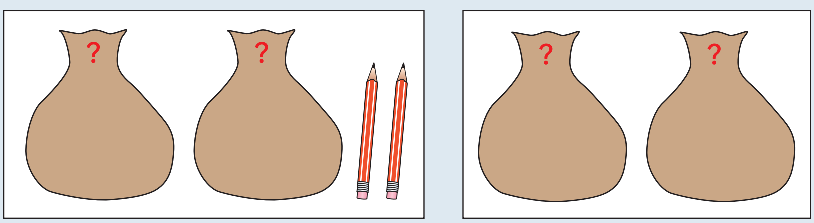
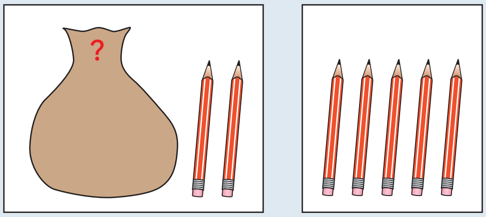
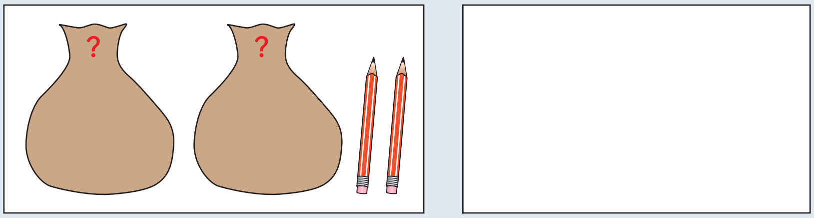
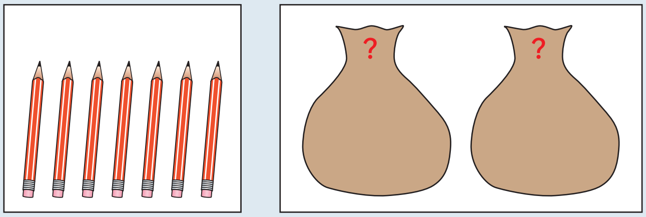


c 

d 

e 

3 Each expression represents the contents of a box of pencils, with x representing the number of pencils in a bag. Draw the contents of each box based on the algebraic expression.

a $x + 2$

b $3 + 4x$

c $(3) + (x + 1)$

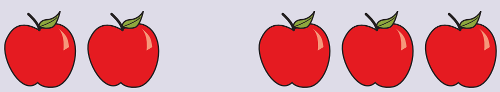
d $(2 + 2x) + (x)$

e $(2 + x) + (2x + 1)$

7.2.2 Simplifying expressions

Explore 7.4

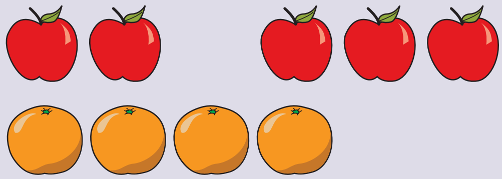
Imagine you have 2 apples in your left hand and 3 apples in your right hand.



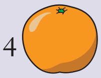
How many fruits do you have in total?

Can you make the expression 2  + 3  simpler?

Now imagine you have 2 apples in your left hand, another 3 apples in your right hand, and 4 oranges in a bag.



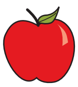
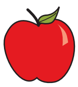
How many fruits do you have in total?

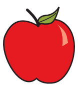
Can you make the expression 2  + 3  + 4  simpler?

Algebraic expressions are sometimes complicated. We can often write them in a simpler, shorter way by performing some operations on the variables.

We know that repeated addition is made shorter by using a multiplication sign. $3 + 3 + 3 + 3 + 3$ can be written as 3×5 or $3 \cdot 5$. In the same way, we can write $x + x + x + x + x$ as $x \times 5$, $x \cdot 5$, or $5x$.

When two terms have the same combination of letters or symbols, we call them **like terms**.

For example, 2  and 3  are like terms, because they both have the same symbol. We have 5  in total.

On the other hand, 3  and 4  are *not* like terms, because the symbols are different.

Look back at the expression from Worked example 7.2: $10x + 10x + 7 \times 10$ Can we simplify this expression?

The expression $10x + 10x + 7 \times 10$ has three terms ($10x$, another $10x$ and 7×10).

We can immediately simplify 7×10 to 70.

$10x$ and $10x$ are like terms, so we can simplify these by adding the 10s: $10x + 10x = 20x$

The simplified expression is now: $20x + 70$

$10x$ and 70 are not like terms because they do not have the same symbol. The expression cannot be simplified any further.

Reminder

Term is the name we give to each quantity that is being added in a sum.

Fact

Adding or subtracting like terms is called **collecting like terms**.

Practice questions 7.2.2

1 Write down the number of terms in each expression.

a $7x + 2$ b $x + 2y + 1$ c $a - b + c$
d $w + 2q$ e $1 - 2x + 3y - 4z$ f $s + 2t - q$

2 Find the like terms in each group.

a $2x, 3, 4x$ b $1, 3w, 4$
c $w, r^2, -4w$ d $2x^2, 4y, -3x^2, y$
e $1, -2s, 4, t$ f $xy, 2x, -3xy, y$
g $2mn, 3n, 4m, -mn, 1$ h $9n^2, 3m, -5mn, 3n^2, 2mn$
i xy, yx, x^2, y, xy^2 j $2st^2, 3t^2, -4, -t^2, st^2, 1$
k $xyz, xy^2, yz^2, -2xyz, -2yz^2$

3 Simplify each expression by collecting like terms.

a $2x + 3x$ b $8y - 4y$ c $a + 2a$
d $3t + 4t$ e $5w - 3w$ f $x + 3x - 2x$
g $x + 3x - 4x$ h $w - 3w + 5w$ i $xy + 2xy$
j $15x - 14x$ k $4ab + ba - 2ab$ l $2x + 3 - x + 4$

4 Simplify each expression by collecting like terms.

a $2x + 3x + y + 2y$ b $-3w + 2t - w + 3t$
c $x + 3 + 3x - 10$ d $2x - x + y$
e $x^2 + 2x^2 + y + 2y$ f $x^2 + 2x^2 + x + 2x$
g $6x + 3 - 4x$ h $4 + t + 5t - 6$
i $10a^2 - 4 - 3a^2 + 5$ j $2x^2 + 3t - x^2 + t$
k $10j + i + i^2 - 5j$ l $-a + b - 3b + a$
m $7a^2 + a - 3a - 4a^2$ n $4ab + a - 3ab + b$
o $x + xy - x^2 + 2xy - x + 2x^2$

5 a Alice buys 3 boxes of chocolates, while Bob buys 5. If there are x chocolates in each box, how many chocolates do Alice and Bob have altogether?

Challenge Q5b

b One kilogram of apples contains n apples. Aishwarya buys 3 kg of apples, but her bag splits and 1 kg is lost. How many apples does she have now?

7.2.3 Evaluating expressions

To evaluate an expression means to find out the value of the expression by replacing the variable with a number and carrying out the calculations. If we evaluate an expression for many values of the variable we can produce a table of values.

In Section 7.2.2, we simplified the expression for the weight, W, of pencils in Worked example 7.2 to $20x + 70$. Complete the table of values for $x = 20$, 30, 40 and 50.

x	20	30	40	50
W	$20 \times 20 + 70 = 470$			

The total weight of the box of pencils was 1570 grams. Look at the values in your table. Can you make an estimate about how many pencils were in each bag?

Algebraic expressions can contain more than one variable. Can you apply the same rules to evaluate expressions when there are many variables?

Connections

You evaluated some expressions for patterns in Chapter 6.

Explore 7.5

At Al's Café, a sandwich costs €2.50, a fizzy drink costs €1.50 and an ice cream costs €2.00. For a party, you need to purchase x sandwiches, y drinks and z ice creams.

Can you write down an expression for A, the total cost of your purchase at Al's Café?

Can you find out how much would you spend if $x = 7$, $y = 12$ and $z = 9$?

At Jodi's Deli, a sandwich costs €2.00, a fizzy drink costs €1.75 and an ice cream costs €2.50.

Can you decide where to shop for the party: Al's Café or Jodi's Deli?

We have to be careful when we evaluate expressions for negative values of variables.

Worked example 7.3

Evaluate $w = 3x - 2y$ when $x = -2$ and $y = -4$

Solution

To prevent mistakes, we can use brackets when we replace variables with numbers:

$w = 3x - 2y = 3 \times (-2) - 2 \times (-4)$

The product of two negative numbers is positive, so the two terms in the expression above become:

$3 \times (-2) = -6$

and

$-2 \times (-4) = 8$

So, $w = -6 + 8 = 2$

Practice questions 7.2.3

1 The table shows the values of the variables a, b, c, x, y and z.

a	b	c	x	y	z
3	−1	0	2	1	4

Evaluate:

a $a + b$ **b** $b - c$ **c** $x^2 - b$

d $3x + a$ **e** $\frac{x}{3} - y$ **f** $-b - 3z$

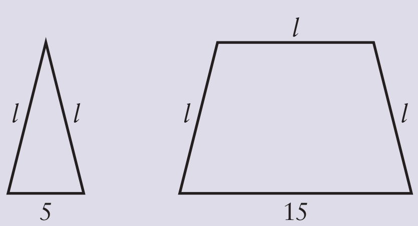
g $\frac{y + x}{a}$ **h** $ax + b$ **i** $ax + by + c$

j $ax + by + cz$ **k** $\frac{cx + bz}{a}$

7.2.4 Brackets and expanding

Explore 7.6

The two shapes below are made with pieces of string. The length of each side, in centimetres, is shown.



Can you write the length of string needed to make each shape using algebraic notation? Use brackets to group the length needed for each shape.

Can you write an expression for the total length of string needed to make both shapes? Do you still need brackets? How can you make the expression simpler?

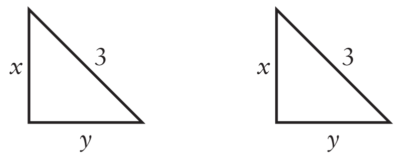
What does 2×5 mean if we write it as repeated addition?

This can be either $2 + 2 + 2 + 2 + 2$ or $5 + 5$

What about $2 \times (5 + 3)$?

This is simply $(5 + 3) + (5 + 3)$ or 2 fives + 2 threes, i.e. $2 \times 5 + 2 \times 3$

Now look at these identical shapes, made with lengths of string.



We can write the length of string needed to make each shape using algebraic notation.

We may write the expressions for the two identical shapes as:

$(2x + 3)$ and $(2x + 3)$

Now, find an expression for the total length of string needed to make both shapes.

By collecting like terms, we can see that:

$(2x + 3) + (2x + 3) = 2x + 3 + 2x + 3 =$ 2 lots of $(2x)$ and 2 lots of 3

$= 2 \times 2x + 2 \times 3$

We can also simplify the expression in a different way by realising that we have two lots of $(2x + 3)$:

$(2x + 3) + (2x + 3) = 2(2x + 3)$

Since the two quantities represent the total length of string:

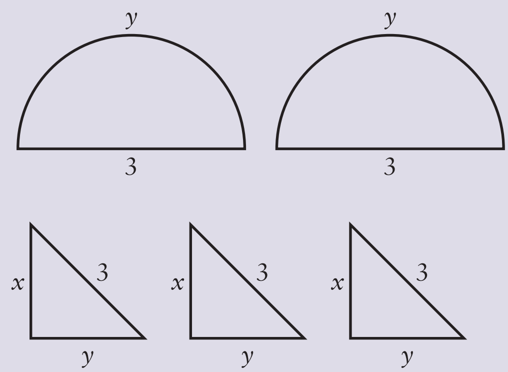
$$2(2x + 3) = 2 \times 2x + 2 \times 3$$
$$= 4x + 6$$

Fact

When we have a quantity multiplied by an expression in brackets, we need to multiply every term in the expression by the quantity outside the brackets. This is called **expanding the brackets**. So: $a(b + c) = ab + ac$.

Explore 7.7

Building on what you have learned, can you find the total length of string required to make the following shapes?



Can you find how much string you would need for four triangles? For 10 triangles? For m triangles?

Fact

To expand brackets, use the formula:

$a(b + c) = ab + ac$

This is called the distributive property of multiplication over addition. This formula works the other way round too. If two terms have a common factor, we can put the common factor outside the brackets:

$ab + ac = a(b + c)$

This process is called **factorising**.

Reminder

Remember, $2x - 4y$ is one way of writing $2x + (-4y)$

Worked example 7.4

a Expand the expression $3(x + 4)$

b Expand the expression $-3(2x - 4y + 5)$

Solution

a We need to remove the brackets from the expression $3(x + 4)$

We can apply the result $a(b + c) = ab + ac$ directly. We multiply the number 3 by both x and 4:

$$\begin{aligned} 3(x + 4) &= 3 \times x + 3 \times 4 \\ &= 3x + 12 \end{aligned}$$

b In this question, we need to be careful about the negative signs. We need to multiply -3 by $2x$ and by $4y$:

$(-3) \times 2x = -6x$, $(-3) \times (-4y) = 12y$ and $(-3) \times 5$

So, $-3(2x - 4y) = -6x + 12y - 15$

Practice questions 7.2.4

1 Write down the number of factors in each expression.

a ab b $2wt$

c $3x(x + 1)$ d $x(x - 2)$

e $ab(c + 1)$ f $(a + c)(a + b)$

2 Expand these expressions.

a $3(b + 3)$ b $2(w + t)$

c $7(x - 1)$ d $(3 - x) \cdot 2$

e $-2(x + 1)$ f $2(-c + 3)$

3 Expand these expressions.

a $3(2b + c + 3)$ b $2(w + 2t - 3)$

c $4(3s + 5t + 1)$ d $-2(2x - y + 1)$

e $(2 + 3x)4$ f $2(3w - 1)$

4 Expand these expressions.

a $b(b + 3)$ b $2t(w + t)$

c $4e(e - 2)$ d $-x(x + 1)$

e $3x(1 - 3x)$ f $-2y(3y + 1)$

5 Expand these expressions, then simplify them by collecting like terms.

a $3(2b + 3) - 4$ b $2(1 + 2t) - t$

c $3 + 4(2m - 1)$ d $x + 2(x - 1)$

e $4(-1 - x) + 2x$ f $4(-y + 2) + 4y$

7.3 Indices

Algebraic expressions with multiplications often involve powers of the same variable. These can also be simplified.

Remember that an expression with a power, such as 3^4, means four 3s multiplied together: $3^4 = 3 \times 3 \times 3 \times 3$

Explore 7.8

Can you tell how many times x is multiplied by itself in the expression x^2? What about in the expression x^5?

Write down $x^2 \times x^5$ using multiplication signs. Can you write the result of $x^2 \times x^5$ as a single power of x?

Can you write these expressions as a single power of the base?

a x^3x^5 **b** a^2a^4 **c** $y \times y^2$

How would you write the product $x^m x^n$ as a single power of x?

Reminder

In a power expression x^a, x is the base.

When a fraction contains a power in the numerator and a power in the denominator, and both have the same base, we can simplify the fraction like this:

$$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x^2}{1} = x^2$$

However, we also have $x^{5-3} = x^2$

Fact

$x^m \cdot x^n = x^{m+n}$

$\frac{x^m}{x^n} = x^{m-n}$

$(x^m)^n = x^{mn}$

When the base of a power is a power itself, we can simplify the expression like this:

$$(x^2)^3 = (x^2) \cdot (x^2) \cdot (x^2) = (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$$

However, we also have $x^{2 \times 3} = x^6$

Practice questions 7.3

1 Write each expression to include a base raised to a single power.

a $x \cdot x \cdot x$ **b** $w \times w \times w \times w \times w$

c $x \cdot x \cdot x \cdot y$ **d** $x \cdot x \cdot y \cdot y$

e $e \times t \times e \times t \times e \times t$ **f** $ab \times ab \times ab$

g $x \cdot x \cdot x \cdot x \cdot y$ **h** $a \times a \times b \times a \times a \times b$

2 Expand these expressions, using multiplication signs instead of powers.

a r^2 **b** x^3 **c** ab^2 **d** $(ab)^2$

e xyz^3 **f** $(xyz)^3$ **g** a^2b^3 **h** $x(yz)^2$

3 Write each expression to include a base raised to a single power.

a r^2r^3 b x^3x c ab^2b^3 d $a(ab)^2$

e $m^2n^2m^3n$ f $3x^2x(2y)$ g $2a(3b)^2a^2$ h $n^2m^3nm^2$

4 Write each expression as a base raised to a single power.

Challenge Q4

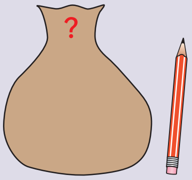
a $\frac{r^3}{r^2}$ b $\frac{x^4}{x^2}$ c $\frac{y^2 \cdot y^3}{y^4}$ d $(x^2)^4$

e $(x^3)^3$ f $t^2 \cdot (t^2)^3$ g $\frac{(x^2)^3}{x^3}$ h $\frac{x \cdot (x^2)^3}{x^2 \cdot x^3}$

7.4 Algebraic equations

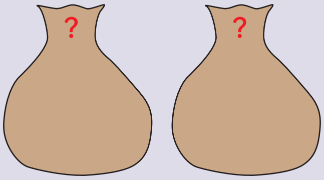
Applied mathematicians use equations to model real-life situations, such as how far the ash will spread when a volcano erupts. By solving the equations, they can make accurate predictions that help people know what action to take.

Explore 7.9



There are some pencils in the bag. If we know that there are 5 pencils in total, how many pencils are in the bag? What is the number x that makes $x + 1$ equal to 5? What is the same about these two questions?

Explore 7.10



Each bag contains the same number of pencils. If we know that there are 10 pencils in total, how many pencils are in each bag? What is the number x that makes $2x$ equal to 10? What is the same about these two questions?

An equation is an expression that contains an equal sign, such as $3 + 5 = 8$ or $20x + 70 = 1570$. Solving an equation means finding the value of the variable that makes the left hand side of the equation equal to the right hand side.

Fact

The expression to the left of the equal sign is called the left hand side (LHS) and the expression to the right of the equal sign is called the right hand side (RHS).

Worked example 7.5

a Solve $x + 3 = 8$ **b** Solve $3x = 12$

Solution

a We need to find the number that we can add to 3 to get 8.

$x + 3 = 8$

The opposite of adding is subtracting. If we subtract 3 from both sides of the equation, we will be left with x on the left hand side.

$x + 3 - 3 = 8 - 3$

We can then evaluate the right hand side to find the value of x.

$x = 5$

If we replace x in the original equation with 5 we get

$5 + 3 = 8$, which is true.

b We need to find the number that we can multiply by 3 to get 12.

$3x = 12$

The opposite of multiplication is division. If we divide both sides of the equation by 3, we will be left with x on the left hand side.

$$\frac{3x}{3} = \frac{12}{3}$$

We can then evaluate the right hand side to find the value of x.

$x = 4$

If we replace x in the original equation with 4 we get

$3 \times 4 = 12$, which is true.

Fact

When we are working with equations, we need to do the same thing to each side of the equation to keep it balanced.

To solve equations that have both additions and multiplications, we need to take two steps.

Worked example 7.6

Solve the equation $3x + 4 = 10$

Solution

$3x + 4 = 10$

We need to carry out operations on both sides of the equation until we are left with x by itself on one side.

Start by subtracting 4 from both sides.

$$3x + 4 - 4 = 10 - 4$$
$$3x = 6$$

Now divide both sides by 3.

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

The solution is $x = 2$

If we replace x in the original equation with 2 we get

$3 \times 2 + 4 = 10$, which is true.

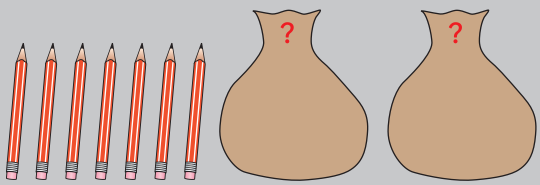
Now we can go back to the situation in Explores 7.9 and 7.10 and find out how many pencils are in each bag.

Worked example 7.7

A box of pencils has been delivered to a classroom.

The pencils have been packaged in two different ways:

- There are seven loose pencils.
- There are two identical bags, each containing the same unknown number of pencils.



The label on the box says, 'weight of one pencil = 10 grams, total shipping weight = 1570 grams'.

How many pencils are in each bag?

Solution

We have already written and simplified an equation to represent the information.

$10x + 10x + 70 = 1570$, where x is the number of pencils in each bag.

As before, we need to carry out operations on both sides of the equation until we are left with x by itself on one side.

First subtract 70 from both sides. Then divide both sides by 20.

$$10x + 10x - 70 = 1570 - 70$$

$$20x = 1500$$

$$\frac{20x}{20} = \frac{1500}{20}$$

$$x = 75$$

So, there are 75 pencils in each bag.

To check that we have solved the equation correctly, we can substitute our value for x back into the equation and evaluate it:

$20 \times 75 + 70 = 1570$

This matches the total weight given for the box, so we can be confident that our answer is correct.

Reflect

What would you do if your answer for the number of pencils was $x = 3.64$ or $x = 124\,000$? Are these reasonable answers in the context of the question?

We can use the same process to solve equations involving subtractions and divisions. Remember that the opposite of subtraction is addition and the opposite of division is multiplication.

What is the number x that makes $x - 1$ equal to 5?

Worked example 7.8

a Solve the equation $x - 3 = 4$ **b** Solve the equation $\frac{x-7}{9} = 2$

Solution

a The opposite of subtraction is addition, so add 3 to both sides.

$$\begin{aligned} x - 3 &= 4 \\ x - 3 + 3 &= 4 + 3 \\ x &= 7 \end{aligned}$$

b The opposite of division is multiplication, so start by multiplying both sides by 9.

$$\begin{aligned} \frac{x-7}{9} &= 2 \\ 9 \times \left(\frac{x-7}{9}\right) &= 9 \times 2 \\ x - 7 &= 18 \\ x - 7 + 7 &= 18 + 7 \\ x &= 25 \end{aligned}$$

Hint

Write each step in your equation neatly. As you get more confident in the process, you might leave out some of the interim steps.

Practice questions 7.4

1 Write out in words the question that corresponds to each equation, then solve it.

a $x + 1 = 3$ b $x - 3 = 6$

c $3 + x = 4$ d $x - 3 = -2$

e $6 - x = 2$ f $4 + x = 6$

g $3x = 12$ h $4x = -16$

i $21 = 7x$ j $-2x = 8$

2 Solve these equations.

a $x + 3 = 6$ b $p + 3 = 4$

c $x + 3 = 10$ d $m + 7 = 15$

e $3 + w = 12$ f $2 + x = 2$

g $15 + d = 18$ h $24 + m = 31$

i $x - 6 = 2$ j $p - 4 = 2$

k $y - 3 = 8$ l $x - 12 = 9$

m $13 - x = 8$ n $4 - x = 2$

o $10 - x = 4$ p $21 - x = 16$

q $3x = 21$ r $2x = 28$

3 Solve these equations.

a $x + 3 = -6$ b $-p + 3 = 4$

c $x - 3 = -10$ d $m - 7 = 15$

e $3 + w = -12$ f $-2 + x = 14$

g $15 + d = -18$ h $-24 + m = -31$

i $x - 6 = -2$ j $p - 4 = -2$

k $-y - 3 = 8$ l $-x - 12 = -9$

m $13 - x = -8$ n $-4 - x = 2$

o $10 - x = -4$ p $-21 - x = 16$

q $3x = -21$ r $-2x = 28$

4 Solve these two-step equations.

a $2x + 1 = 9$ b $2x + 1 = 10$ c $3x - 1 = 11$

d $2x + 5 = 23$ e $9 - 2x = 11$ f $5x - 10 = 10$

g $3x + 4 = 4$ h $24 - 4x = 0$ i $6x - 10 = 14$

5 Solve these equations.

a $x + 1.3 = 4.4$ b $x - 3.2 = -3.4$ c $4.3 - x = 1$

d $1.2x = 3.6$ e $5x = 37.5$ f $1.3x = 13$

g $x + 1.4 = 0.8$ h $2x - 1 = 1.5$ i $1.5x + 3 = 1.5$

j $\frac{x}{2} + 1 = 2$ k $3 - \frac{x}{3} = 2$ l $\frac{x}{2} - \frac{3}{2} = 5$

6 Expand the brackets then solve these equations.

a $3(x + 1) = 6$ b $2(1 - x) = 4$

c $2(1 - x) = 5$ d $10(3 + x) - 9x = 31$

e $x + 3(x - 1) = 0$ f $3 + 8(2x - 1) - 14x = 2$

g $2 - (x + 3) = 4x + 3 - 4(x - 1)$ h $x - (x - 3) = 2x + 3$

Hint 6g

$2 - (x - 3)$ means $2 + (-1) \times (x - 3)$

7.5 Problem-solving skills

7.5.1 Word problems

Explore 7.11

Some birds migrate every autumn from Iceland to West Africa, a distance of 6000 km. Geolocators used to track the migration of such birds show that they can complete the non-stop sea crossing flying at estimated speeds of 50 km per hour.

Can you set up an equation that helps you find an estimate of how long it takes such birds to fly this distance?

In this section, we are going to look at how algebra can be used as a problem-solving tool.

Worked example 7.9

A brick weighs a kilogram plus half the weight of the brick. What is the weight of the brick?

Solution

In this question, the unknown quantity is the weight of the brick. We can use the variable w to represent this quantity.

We have been given only one piece of information: the weight of the brick equals one kilogram plus half the weight of the brick.

We can convert this statement into an equation and then solve it.

the weight of the brick	w
equals	$=$
one kilogram	1
plus	$+$
half the weight of the brick	$\frac{1}{2}w$

So, our equation is $w = 1 + \frac{1}{2}w$

Now we need to solve this equation.

There is a w on both sides of this equation. We do not yet have a method for solving an equation with an unknown value on both sides, but we can use a table to evaluate both sides for different values of w.

LHS, w	0	1	2	...
RHS, $1 + \frac{1}{2}w$	1	1.5	2	...

The LHS equals the RHS when $w = 2$, so this is the solution to our problem: the brick weighs 2 kilograms.

Two kilograms is a reasonable weight for a brick, so our answer makes sense in the context of the question.

Reflect

Look at the two different wordings used in Worked example 7.9.

- A brick weighs a kilogram plus half the weight of the brick.
- The weight of the brick equals one kilogram plus half the weight of the brick.

What is the same about these sentences? What is different? Which one is easier to understand? Which one is easier to change into an equation?

We can apply these steps to other word problems.

1. Reword the given information so it contains the word ‘equals’.
2. Introduce a variable for the unknown quantity.
3. Convert the given information into expressions for the left and the right hand sides of an equation.
4. Set up and solve the equation.
5. Check that our answer is reasonable.

Practice questions 7.5.1

1 Choose a suitable variable or variables and convert each of these into an algebraic expression.
  - a three times the mass
  - b 4 metres more than the perimeter
  - c the sum of two consecutive numbers
  - d twice Annalisa's age
  - e two years older than Fred
  - f one year younger than Bill
  - g the difference between the base and the height
  - h one half of the product of base and height
  - i one half of the sum of the two diagonals.

Hint Q1c

Consecutive numbers come one after the other. For instance, 13 and 14 are consecutive numbers because 14 is the next number after 13.

2 Find the number that satisfies each statement by writing and solving an equation.
  - a A number plus 4 is 10.
  - b Subtracting 5 from a number gives 7.
  - c The difference between a number and 4 is 1.
  - d Three times a number is 12.
  - e A number divided by 4 is 6.
  - f Twenty divided by a number is 5.
  - g A number plus 5 is −3.
  - h Five minus a number is 0.
  - i The difference between 4 and a number is −1.
  - j Five times a number is 2.
  - k A number divided by 3 is −4.
  - l Negative ten divided by a number is 5.

3 Find the number that satisfies each statement by writing and solving a two-step equation.
  - a The difference between three times a number and 8 is 1.
  - b Four plus twice a number is 8.
  - c Dividing a number by three and adding 8 gives 10.
  - d Subtracting four times a number from 50 gives 30.

4 Solve each problem by writing and solving an equation.
  - a A set of notebooks costs \$32. Each notebook costs \$4. How many notebooks are in the set?
  - b Another set of notebooks costs \$25. There are 5 notebooks. How much does each notebook cost?

c Three kilograms of apples and two kilograms of pears cost €12. One kilogram of pears costs €3. How much does one kilogram of apples cost?

d The sum of two consecutive whole numbers is 11. Find the two numbers.

e The sum of two consecutive even numbers is 22. Find the two numbers.

f The sum of three consecutive odd numbers is 39. Find the three numbers.

g Three sandwiches and two drinks cost \$12.00. A sandwich costs twice as much as a drink. Find the cost of a drink.

h Three sandwiches and two drinks cost \$12.00. A sandwich costs \$0.50 more than a drink. Find the cost of a drink.

5 A supercomputer requires 24 modules of RAM to operate. Each RAM module is 20 gigabytes. How many 16 gigabyte modules could be used instead?

7.5.2 Graphical solutions

Explore 7.12

Construct a table of values for $y = x + 5$ for $x = 0, 1, 2, 3, 4$. Plot your values on a graph.

Construct a table of values for $y = 7 - x$ for $x = 0, 1, 2, 3, 4$. Plot your values on the same graph.

What do you notice about the points you have plotted? What does this tell you about the equation $x + 5 = x + 7$?

Connections

You learned how to draw graphs of rules in Chapter 6.

Graphs can be a powerful method for solving equations.

Worked example 7.10

The sum of three consecutive whole numbers is twice the largest number. Find the smallest number of the three.

Solution

We can follow the steps from the end of Section 7.5.1.

1 Reword the given information so it contains the word ‘equals’.
The sum of three consecutive whole numbers equals twice the largest number.

2 Introduce a variable for the unknown quantity.
The value we need to find is the smallest number of the three. We will call this number x.
We do not need to choose variables for the other two numbers, because we can express these numbers in terms of x. The next consecutive whole number is 1 more than x, so it is $x + 1$, and the third is $x + 2$.

3 Convert the given information into expressions for the left and the right hand side of an equation.
On the left hand side, we have 'the sum of three consecutive whole numbers'. This is:

$$x + (x + 1) + (x + 2)$$

We can simplify this to $3x + 3$
On the right hand side, we have 'twice the largest number'. This is:

$$2(x + 2)$$

We can expand the brackets to get $2x + 4$

4 Set up and solve the equation.
To set up the equation, we put an equals sign between the left hand side and the right hand side.

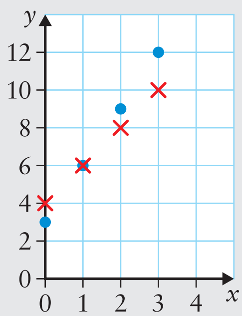
$$3x + 3 = 2x + 4$$

We can use a table of values to solve an equation like this. Another method is to use graphs.
We start by constructing a table of values for each side of the equation, for the same values of x.

x	0	1	2	3	...
LHS, $3x + 3$	3	6	9	12	...

x	0	1	2	3	...
RHS, $2x + 4$	4	6	8	10	...

Now we can plot these points on the same graph. We use dots for the LHS points and crosses for the RHS points, so that we can tell them apart.



Looking at the graph, there is one point that has both a dot and a cross: (1,6). This is the point where $x = 1$ and $y = 6$
This means that when $x = 1$, both the left hand side and the right hand side of the equation are equal to 6. So, the solution to $3x + 3 = 2x + 4$ is $x = 1$
The smallest number of the three consecutive whole numbers is 1.

5 Check that our answer is reasonable.

If 1 is the smallest of the three numbers, then the other two are 2 and 3, since they are consecutive.

The sum of the three numbers is $1 + 2 + 3 = 6$

Twice the largest number is $2 \times 3 = 6$

These results are the same, so this confirms that our answer is correct.

Reflect

Which method do you prefer: constructing a table of values or drawing a graph? Can the graphical method be used for any equation? What steps do you need to follow to use the graphical method?

Investigation 7.3

There are many graphing software packages available for free online. With the help of your teacher, librarian or guardian, find some of these packages and learn how to use them.

Use a software package to solve the equations $\frac{1}{x} = x$ and $x^2 = x$

Research skills

Practice questions 7.5.2

1 Create a table of values for both the LHS and the RHS of each equation. Then graph both the LHS and the RHS on the same Cartesian plane and estimate a solution to the equation.

- **a** $3x + 1 = 9 - x$
- **b** $3 - 2x = x - 6$
- **c** $x + 5 = 10 - x$
- **d** $3(x + 1) = -2(x - 9)$
- **e** $2x - 1 + 3(x - 1) = 2x + 2$
- **f** $x + 5 = 10$
- **g** $0 = -3 + x$
- **h** $2(x + 3) = 3x + 6$
- **i** $3 + 2(x - 4) = x - 5$
- **j** $3x - 1 = -x - 9$
- **k** $x + 1 = -5 - x$
- **l** $3(x - 4) + 1 = 2(x + 1) + x$
- **m** $3(x - 4) + 14 = 2(x + 1) + x$

2 **a** Plot graphs for each of the following equations for $x = 0, 1, 2, 3, 4$.

- **i** $2(x - 1) = 2x + 2$
- **ii** $3x + 1 = 3(x - 4) + 13$

b Describe how many solutions there are to each of the equations in part **a**.

Hint Q1j

The solution can be negative. Use the trend shown by the graph to choose which x values you need for the table of values.

Hint Q2

When no x value makes the LHS equal to the RHS, the equation has no solution. In this case, we say that the equation is inconsistent or unsolvable. When any x value makes the LHS equal to the RHS, the equation has infinitely many solutions. In this case, we say that the equation is indeterminate.

7 Algebra and equations

Self assessment

- I can use the conventions and abbreviations used for algebraic expressions.
- I can distinguish between terms and factors.
- I can use index notation.
- I can recognise and use grouping symbols.
- I can evaluate and simplify algebraic expressions.
- I can recognise the meaning of algebraic expressions in context.
- I can expand expressions with brackets, for both positive and negative numbers.
- I can simplify products of powers.
- I know what solving an equation means.
- I can solve linear equations in one and two steps.
- I can convert a word problem into an equation.
- I can solve linear equations graphically.

Check your knowledge questions

1 Expand the brackets, then simplify each algebraic expression.

a $3(x-1)+4(x+5)$ **b** $2(1-x)-4(x+4)$

c $4(2x-1)+3(3-x)$ **d** $-3(x-2)+3(3-2x)$

e $-2(x-1)+3(-x+3)$ **f** $2(x-2)-2(x-2)$

g $a-b+3a+4b$ **h** $a(3+b)-b(2+a)$

i $xy-3x(y+1)-x$ **j** $x^2+3x-x(x+1)$

k $x(x+1)-2x(2x-1)$ **l** $ax(x+y)-x(ax-ay)$

2 Evaluate each expression when $x = 1$, $y = 2$, $a = 0$ and $b = -1$

a $3x+y$ **b** x^2-3y **c** $a+b+2x$

d $xy+3$ **e** $\frac{y}{x}+b$ **f** $ax+y$

g $\frac{x+y}{3}-a$ **h** a^2+bxy **i** $b(x+y)+b$

j $a(x-b)+y^2$ **k** $\frac{bx-y}{y}-\frac{ay-x}{x}$ **l** $x\frac{b}{y}+b\frac{x}{y}$

3 Solve each equation.

a $x-2=3$ **b** $r+4=12$ **c** $3+w=5$ **d** $10=x-7$

e $4=7-x$ **f** $3+x=5$ **g** $3+x=3$ **h** $3+x=1$

i $3+x=0$ **j** $-x-4=3$

4 Solve each equation.

a $3x=12$ **b** $4x=8$ **c** $15=5x$

d $\frac{x}{2}=8$ **e** $\frac{x}{3}=4$ **f** $\frac{x}{6}=5$

g $2x=-6$ **h** $-2x=10$ **i** $-x=10$

j $\frac{x}{2} = -4$ k $\frac{x}{-3} = 7$ l $\frac{x}{-2} = -4$

m $\frac{21}{x} = 7$ n $\frac{16}{x} = -4$ o $\frac{14}{-x} = 7$

5 Simplify and solve each equation.

a $3x + 1 = 13$ b $-3 + 2x = 9$

c $2x - 1 = -9$ d $4x - 1 = -5$

e $3(x + 1) - x = 11$ f $x + 2(x + 3) = 3$

g $10 = 3x + 4(x - 1)$ h $3x - 2(x - 1) = 4$

6 An equilateral triangle has sides of length x. The perimeter of the triangle is 24 cm. Find the lengths of its sides.

7 The base of a rectangle is 12 cm longer than its height. The perimeter of the rectangle is 60 cm. Find the base of the rectangle.

8 The base of a rectangle is twice the height. The perimeter of the rectangle is 60 cm. Find the base and the height of the rectangle.

9 The sum of two consecutive whole numbers is 55. Find the two numbers.

10 Frank's phone costs 1.2 times as much as Andrew's phone. The total value of Frank and Andrew's phones is $220. How much does Andrew's phone cost?

11 The amount of money M (in dollars) earned by Jacques when he works for x hours is given by the rule $M = 3x + 5$

a How much does Jacques earn when he has worked for 4 hours?

b Jacques wants to buy a new T-shirt that costs $35. How many hours does he have to work to earn enough money?

12 A school bus company charges its customers £50 a year for the subscription to its service. Each trip to and from the school costs an additional £0.50.

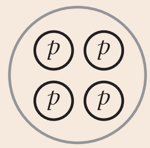
a Carolina uses the bus a total of 234 times. How much does the company charge her?

b The company charges Laura £153. How many times did she use the bus?

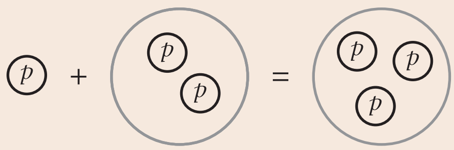
13 A class of 30 students is planning on buying a present for their teacher's birthday, but only 25 students want to participate in the present. If all the students in the class participated, the contribution would be \$5.00 each.
Find how much each of the 25 students has to pay.

14 The nuclei of different elements on the periodic table contain a different number of protons. Hydrogen has one, helium has two, lithium has three and beryllium has four.

p represents a single proton, so a beryllium nucleus can be drawn like this:

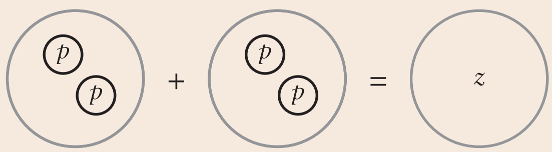


In nuclear reactions, nuclei are split and the protons in them are recombined in different arrangements: a proton and a helium nucleus could combine and give a lithium nucleus. This reaction is represented as:

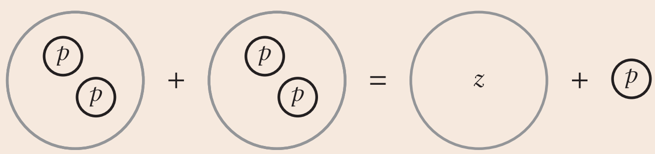


For each reaction below, find the name of the nucleus with the unknown number, z, of protons.

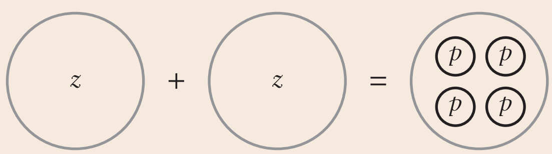
a



b



c



15 Solve the following equations by graphing both the left and the right hand sides on the same Cartesian plane.

a $x + 1 = 3 - x$ **b** $4 - x = x + 2$ **c** $2x - 4 = 5 - x$

d $5 + x = 2x$ **e** $5 + x = -2x - 1$ **f** $2(x + 4) = 3x + 8$