Extending Children's Mathematics

Fractions and Decimals

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Foreword by Thomas P. Carpenter

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To our children Eric and Nick (SBE), and Kevin and Eve (LL)

Our love for you inspires us to do our part to make schools better for all children.

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chapter 4

Relational Thinking

Connecting Fractions and Algebra

In this chapter, we step back to examine the deeper structures of children's mathematical thinking. A small set of mathematical relationships governs how numbers, operations, and equations work in arithmetic as well as in algebra. We call these relationships the *fundamental properties of operations and equality*. When children are encouraged to use their own strategies to solve problems involving fractions, they intuitively draw on these fundamental properties. When teachers encourage such strategies and create opportunities for students to reflect on the mathematics relationships that they used, learning fractions serves as a foundation for learning algebra with understanding. We call the strategies in which children draw upon these fundamental properties, whether implicitly or explicitly, *Relational Thinking* strategies

To begin to use Relational Thinking strategies, children need to understand

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fractional quantities as relational. We begin the chapter by describing what this understanding is and how Equal Sharing and Multiple Groups problems support its development. We then introduce Relational Thinking and revisit children's strategies for Multiple Groups problems to highlight when and how they involve Relational Thinking. We conclude by discussing how Relational Thinking enables students to understand the connections between learning arithmetic and algebra.

Relational Understanding of Fractions

Students who can express a number in terms of other numbers and operations on those numbers hold a *relational understanding of the number*. Understanding numbers relationally helps students use mathematical relationships to solve problems. For example, a child who understands that 5 can be decomposed into 2 and 3 can use that understanding to solve $8 + 5 = n$ by first adding $8 + 2$ to get 10 and then adding 3 more to get $n = 13$. Similarly, a child who understands that 45 can be thought of as 4 tens and 5 ones, and 36 as 3 tens and 6 ones, can use this understanding to add 45 and 36 by combining tens, combining ones, and then combining the results: $40 + 30 = 70$; $5 + 6 = 11$; so $45 + 36 = 70 + 11 = 81$. In both of these examples, students used an understanding of how one amount could be expressed in terms of other amounts to simplify the problem and facilitate a solution.

As children solve Equal Sharing problems, they deal with two distinct types of relationships that are essential to a *relational understanding of fractions*. First, children learn to relate the process of partitioning a whole unit into n equal parts with the size of a part, $\frac{1}{n}$, that results. We call this understanding a *relational understanding of unit fractions*. Second, children learn that they can combine unit fractions to make *fractions that are multiples of unit fractions*, such as $\frac{3}{4}$ and $\frac{5}{4}$, as well as mixed numbers, such as $1\frac{7}{8}$. We refer to this understanding as a *relational understanding of fractions as composite*.

Relational Understanding of Unit Fractions

A unit fraction is any fraction that has the form $\frac{1}{n}$ (where n is any positive whole number not equal to 0). A unit fraction is defined by its relationship to the whole and the two interconnected ideas that 1 unit can be divided into any number of equal parts *and* those parts can be recombined to make 1 again. For example, if n

people are sharing 1 thing equally and completely, each person gets a share that is exactly $1/n$ of that thing:

$$1 \div n = 1/n$$

If all of the shares are recombined, the whole thing is reconstituted:

$$n \times 1/n = 1$$

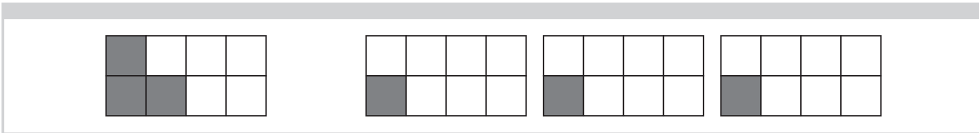
The multiplicative relationship between a part and its whole is reversible in the sense that the whole can be broken apart into unit fractions and the unit fractions can be put back together to make the whole. This essential understanding becomes abbreviated for children as:

$$n/n = 1$$

We described in Chapter 1 how young children tend to separate the process of creating equal shares ($1 \div n$) from the resulting fractional share ($1/n$). They may say, as one child did, "I split the candy bar in threes [i.e., into 3 equal parts] and gave each person a half." As children learn to coordinate the process of partitioning with the result of the partitioning—"I split this candy bar into 3 equal parts and gave each person a third of it"—they begin to think flexibly of unit fractions in terms of a reversible relationship with the whole unit. They think of the whole as a unit that can be partitioned. You can help children express that relationship as "1 split into 3 equal groups is $1/3$" and "3 groups of $1/3$ is 1." As children reflect on these relationships when they use them in their solutions to Multiple Groups problems, these different ways of thinking about thirds become consolidated in the relationship "$3/3$ is the same as 1." You can assist this development by listening closely as children explain their reasoning and by providing mathematical phrasing or number sentences to represent these emergent relationships.

Relational Understanding of Fractions as Composite

Equal Sharing problems provide a natural context for children to combine unit-fraction quantities. They learn in the context of solving problems and representing their solutions that any fractional share m/n can be expressed in terms of unit-fraction amounts. These unit-fraction amounts do not need to come from the same whole—only from a set of wholes that are equal in size. For example, $3/8$ of 1 candy bar is the same as $1/8$ of each of 3 candy bars (Figure 4–1).



$3/8 \quad = \quad 1/8 \quad + \quad 1/8 \quad + \quad 1/8$

Figure 4–1. *$3/8$ of 1 is the same as $1/8$ of 1 three times.*

In other words, Equal Sharing helps children begin to understand the fraction m/n in terms of the relationships between a whole unit, that unit segmented into equal parts, and some combination of those parts from different units of the same size. Children first think of this composite relationship as additive and later as multiplicative:

$$\underbrace{1/n + 1/n + 1/n + \ldots + 1/n}_{m \text{ times}} = m/n$$

$$m \times 1/n = m/n$$

Students might state this idea as, "*m* groups of one-*n*th is the same as *m n*ths." There is no limit to the value of m in m/n. The relationship holds equally well for $4/17$ and $17/4$. Solving problems that involve two or more wholes, as recommended in Chapter 1, helps children generalize this relationship.

As children's understanding of these fraction relationships grows, they begin to use their understanding in combination with the fundamental properties of operations and equality to think relationally about adding, subtracting, multiplying, and dividing fractions.

Introduction to Relational Thinking

Characterizing Relational Thinking

When children use *Relational Thinking* to solve problems, they are drawing upon a small set of fundamental properties that govern how operations and equations work

(Carpenter et al. 2003; Empson et al. in press). Children's strategies often reflect an intuitive understanding of these properties as they use them to enact solutions and to structure their thinking.

Adults, like children, often use Relational Thinking intuitively. We invite you to stop for a moment to think about how you would figure the total amount of money spent on 5 stuffed bears each costing $14 without writing anything down.

A third grader reasoned that 5 groups of $10 would be $50, and then 5 groups of $4 would be $20, for a total of $50 + $20 = $70 for the bears. If you reasoned similarly, then you used Relational Thinking. One way to represent the logic underlying this thinking is:

$$5 \times 14 = (5 \times 10) + (5 \times 4)$$

That is, 5 groups of 14 can be transformed into 5 groups of 10 plus 5 groups of 4. The fundamental property represented by this equation is the *distributive property of multiplication over addition* (often simply called "the distributive property"). The third grader had never heard of this property, nor had he been explicitly taught to use this property to multiply. Instead, his understanding of base-ten numbers and how multiplication worked guided his thinking about this calculation.

Other students may use different types of Relational Thinking to solve the same problem. For example, a sixth grader solved the problem and explained, "When I have to multiply a number by 5, I just multiply half of the number by 10, so 5 times 14 is 70." A way to represent his thinking is:

$$5 \times 14 = 5 \times (2 \times 7) = (5 \times 2) \times 7 = 10 \times 7 = 70$$

The fundamental property represented by this equation is the *associative property of multiplication*. Like the third grader, the sixth grader had not been explicitly taught to use this property to multiply. His understanding about how multiplication works guided his thinking about this calculation.[1]

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[1] For more examples of how children's strategies for multiplication and division of whole numbers are based on these properties, see Baek, 2008.

Children use Relational Thinking in their solutions to whole-number and fraction problems before they learn to write equations to represent the relationships or use conventional terms to describe their thinking. Equations such as $5 \times 14 = (5 \times 10) + (5 \times 4)$ help us identify and represent the logic behind a child's thinking. Conventional terms such as *the distributive property* help us describe and classify this logic. In this chapter, we use equations and conventional terms to represent the deep structure of children's thinking and the kinds of explicit understanding students can attain. The way we use equations and terms in this chapter differs in some ways from how we use equations and conventional terms with children. In Chapter 5, we discuss how and when you can introduce equations to children and use them in instruction.

Because children's understanding of these fundamental properties of operations and equality is contained in how they *relate* one numerical expression—such as 5×14—with another—such as $(5 \times 10) + (5 \times 4)$—we call it *Relational Thinking*. To see the possibility of this relationship in 5×14 requires both a generalized understanding of how multiplication and addition are related and the use of this understanding to guide the solution for this specific computation. When children solve problems involving fractions, they draw upon the same fundamental properties combined with their understanding of specific fraction relationships.

Relational Thinking and Fractions

Consider what a group of first graders was able to do without having been taught a procedure for adding fractions with unlike denominators (Empson 1999)—a topic that appears much later in the elementary curriculum. At the conclusion of five weeks of instruction that focused on solving and discussing Equal Sharing problems, they were given this story problem, a type of problem they had never solved before:

> TINA AND TONY PAINTED PICTURES this afternoon. Tina used half a jar of blue paint for her picture. Tony used three-fourths of the same size jar of blue paint for his picture. How much blue paint did Tina and Tony use altogether for their paintings?

To combine ½ and ¾, 8 out of the 17 first graders thought of ¾ as equal to ½ + ¼ and reasoned that ½ plus another ½ was equal to 1, and then plus another ¼ was 1¼. These children could not have used this strategy if they had not been able to think of ¾ as an amount that was equal to ½ + ¼. The logic of the children's thinking can be represented by the equations in Figure 4–2.

The first graders worked from what they understood to solve a new type of problem. They did not need to be shown a procedure to combine these two fractions because they understood how to decompose ¾ and figured out how to use this decomposition to help them add ½ + ¾. This strategy involved the implicit use of the associative property of addition, which states that if three numbers are to be added, either the first and second numbers or the second and third numbers can be

► We provide tables such as Figure 4–2 to highlight the deep structure of children's thinking and the kinds of explicit understanding of mathematics that students can attain. Although we encourage you to spend some time connecting children's strategies with equations and properties, we want you to know that you do not need to make these connections right now to continue reading this book or experimenting with the ideas in it. Some readers may wish to attend only to descriptions of children's strategies at this point and some may wish to concentrate on the fundamental properties in children's reasoning. Either way is fine. The important thing is to try some problems with your students and study their strategies. With experience, you will find it easier to make connections between children's strategies, equations, and the fundamental properties of operations and equality.

Children's Thinking	Possible Equation to Represent Children's Thinking	Fundamental Property That Is Basis of Children's Thinking
"¾ has ½ and ¼ in it."	¾ = ½ + ¼ ½ + ¾ = ½ + (½ + ¼)	Relational understanding ¾ as equal to ½ + ¼
"½ and ½ make a whole."	½ + (½ + ¼) = (½ + ½) + ¼	Associative property of addition
"A whole plus 1 extra fourth is 1 and 1 fourth."	(½ + ½) + ¼ = 1 + ¼ = 1¼	Relational understanding of 1 as equal to ½ + ½

Figure 4–2. *Using Relational Thinking to add fractions*

added first. It is an example of how fraction instruction can help students develop their understanding of mathematical relationships and how a focus on mathematical relationships can help students learn fractions with understanding.

Younger children's emergent use of Relational Thinking is remarkable in light of the fact that older students often have difficulty learning to explicitly recognize and apply the very same fundamental properties in their solutions to algebraic equations. For example, when we taught high school algebra, many of our students struggled to apply the distributive property to multiply 4 by the sum of x and y in the expression $4(x + y)$. They would incorrectly conclude that $4(x + y) = 4x + y$ rather than $4(x + y) = 4x + 4y$. Similarly, students would inappropriately apply the associative and commutative properties to subtraction and conclude that $8x – (8 – 4x)$ was equal to $4x – 8$ rather than $12x – 8$. Cultivating children's use of Relational Thinking prepares them to understand algebraic concepts and manipulations such as these.

Relational Thinking emerges early in children's thinking about whole numbers and fractions. Most children use some form of Relational Thinking to solve addition and subtraction problems by the end of second grade, even in classrooms where the teacher does not focus on it. But without teacher guidance, children do not continue to develop their capacity to think relationally. In many cases, it atrophies, as it seemed to have done for our high school students, and students abandon making sense of mathematics.

In the following section, we revisit children's solution strategies for Multiple Groups problems to draw your attention to the emergence and use of Relational Thinking.

Thinking Relationally About Multiple Groups Problems

As children come to understand fractions as relational, they begin to use fundamental properties of operations and equality to reason about operations and computations involving fractions. In Chapters 1 and 3, we introduced the types of strategies children use to solve Multiple Groups problems. In this section, we focus on the deep structure of these strategies by examining how children's strategies depend upon the use of Relational Thinking. We distinguish between the type of strategy that a child uses and how that strategy draws on Relational Thinking. For example, a child may use a grouping strategy to solve a problem, but the way the child forms groupings and combines them can reflect the use of different fundamental properties of operations and equality—that is, different forms of Relational Thinking. Children's use of Relational Thinking emerges as they begin to use grouping strategies to simplify their calculations and continues to develop as their strategies become more sophisticated.

Relational Thinking in Children's Grouping and Combining Strategies

Cam solved a Multiple Groups multiplication problem involving 15 groups of ⅔ of a yard of fabric by combining groupings that related 3 pillows and 2 yards (see Figure 3–7). Her work is represented in the first two columns of the table in Figure 4–3. Each line of Cam's table represents a relationship between the number of pillows, the amount of fabric needed for 1 pillow, and the total amount of fabric used for the number of pillows. We show these relationships in the third column using equations. Cam did not write equations, but she understood that she was working with the relationship between groupings of pillows and the total amount of fabric needed for the

Cam's Work		Equation
Yards	***Pillows***	
⅔	1	$1 \times \frac{2}{3} = \frac{2}{3}$
1⅓	2	$2 \times \frac{2}{3} = 1\frac{1}{3}$
2	3	$3 \times \frac{2}{3} = 2$
4	6	$6 \times \frac{2}{3} = 4$
6	9	$9 \times \frac{2}{3} = 6$
8	12	$12 \times \frac{2}{3} = 8$
10	15	$15 \times \frac{2}{3} = 10$

Figure 4–3. *Equations to represent the relationships between amount of fabric and number of pillows in Cam's table*

grouping. (Before looking at the entire table you might cover the rightmost column and try to write an equation that represents the relationship between the number of pillows and amount of fabric used.)

The way Cam used her table to solve the problem shows an *implicit* understanding of the distributive property. For example, she combined 4 yards for 6 pillows with 2 yards for 3 pillows to get 6 yards for 9 pillows. Each of those groupings involves a multiple of $\frac{2}{3}$—the amount of fabric for a single pillow—and so combining groupings involves combining multiples of $\frac{2}{3}$. 6 groups of $\frac{2}{3}$ combined with 3 more groups of $\frac{2}{3}$ is the same as 9 groups of $\frac{2}{3}$ (Figure 4–4). Because 6 groups of $\frac{2}{3}$ is equal to 4 and 3 groups of $\frac{2}{3}$ is equal to 2, 9 groups of $\frac{2}{3}$ is equal to 4 + 2 or 6:

$$9 \times \frac{2}{3} = 6$$

Cam applied this property repeatedly throughout her strategy whenever she combined another 3 groups of 2.

Cam knew that she did not need to represent each fractional amount individually. She created more efficient groupings of fractional amounts and then combined the groupings on the basis of fundamental properties of operations and equality.

Cam's Thinking	Possible Equation to Represent Cam's Thinking	Fundamental Property That Is Basis of Cam's Thinking
"If 6 pillows can be made with 4 yards, and 3 pillows can be made with 2 yards, then 9 pillows can be made with 6 yards."	$(6 \times \frac{2}{3}) + (3 \times \frac{2}{3}) = (6 + 3) \times \frac{2}{3}$ $= 9 \times \frac{2}{3}$	*Distributive property*

Figure 4–4. *Using Relational Thinking to solve a Measurement Division problem*

Relational Thinking in Multiplicative Strategies

Trenton solved a Measurement Division problem that involved finding how many groups of $1\frac{1}{2}$ cups were in 12 cups of frog food (Figure 4–5). His goal was to count the number of one and one-halves in 12.

He began building up from $1\frac{1}{2}$ by doubling $1\frac{1}{2}$ using the distributive property. He said that 2 groups of $1\frac{1}{2}$ could be figured by multiplying 2×1 and then $2 \times \frac{1}{2}$ and adding them together.

Next he drew upon the associative property to relate that grouping, 2 groups of $1\frac{1}{2}$ cups is 3 cups, to the total amount of frog food, 12 cups, in the following way. He doubled $1\frac{1}{2}$ to make 3 cups, and then multiplied that by 4 to make 12 cups. To keep track of the number of one and one-halves he had used to build up to 12, he had to keep track of the relationship between $1\frac{1}{2}$ cups and 12 cups as he related 3 cups for 2 days to 12 cups for 8 days. Because multiplication is associative, he was able to regroup 4 groups of $2 \times 1\frac{1}{2}$ into 4×2 groups of $1\frac{1}{2}$. Figure 4–5 shows each step of Trenton's strategy, equations that could be used to represent his thinking, and the fundamental properties that Trenton drew upon.

Kylie used a basic fraction relationship in her solution that many children come to use in multiplying and dividing fractions (see pp. 60–61 in Chapter 3). She wanted to figure how many groups of $\frac{3}{8}$ were in $10\frac{1}{2}$. She began with the knowledge that 8 groups of $\frac{3}{8}$ is 3:

$$8 \times \frac{3}{8} = 3$$

Trenton's Thinking	Possible Equation to Represent Trenton's Thinking	Fundamental Property That Is Basis of Trenton's Thinking
"2 days uses 3 cups of food."	$2 \times 1\frac{1}{2} = (2 \times 1) + (2 \times \frac{1}{2})$ $= 2 + 1$ $= 3$	Distributive property Fractions as multiples of units fractions
"8 days uses 12 cups of food, because if I multiply 2 days by 4 to get 8 days I need to multiply 3 cups by 4 to get 12 cups."	$4 \times (2 \times 1\frac{1}{2}) = 4 \times 3$ $(4 \times 2) \times 1\frac{1}{2} = 4 \times 3$	Associative property
	$8 \times 1\frac{1}{2} = 12$	Multiplication

Figure 4–5. *Relational Thinking in Trenton's strategy*

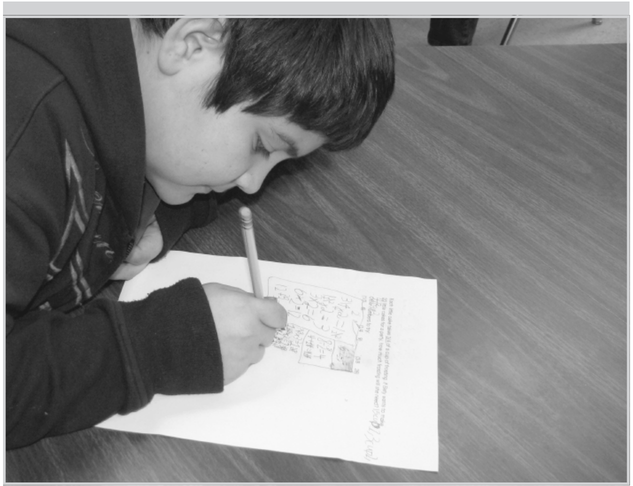

Figure 4–6. Thinking relationally

When we asked her how she knew that 8 groups of $3/8$ was 8, she said, "I just know that if I multiply a fraction by the denominator it equals the number in the numerator. One way to show that this works is to think 8 groups of $1/8$ is 1, so 8 groups of $3/8$ will be 3." The relationship $b \times a/b = a$ is true for all fractions, a/b, as long as $b \neq 0$. Many students justify it on the basis of the commutative and associative properties of multiplication as Kylie did. Figure 4–7 represents this reasoning with equations and the fundamental properties that justify the relationships from one equation to the next. We invite you to identify how the rest of Kylie's strategy embodied Relational Thinking.

Julie's strategy for figuring 15 groups of $2/3$ yard of fabric each (p. 59) drew on the associative property of multiplication. She said that 15 groups of $2/3$ was the same as 30 groups of $1/3$, which was equal to 10 yards. Her thinking is represented in Figure 4–8.

Some of the most sophisticated Relational Thinking strategies for Multiple Groups problems involve reasoning about equations. For example, a sixth grader was given this problem:

> EACH LITTLE CAKE TAKES $3/4$ of a cup of frosting. If Bety wants to make 20 little cakes for a party, how much frosting will she need?

He immediately recognized the situation as multiplication and wrote:

$$20 \times 3/4 = p$$

He said, "I can halve the number of groups and double the size of each group," and wrote:

$$20 \times 3/4 = 10 \times 1\tfrac{1}{2}$$

Possible Equation	Fundamental Property That Is Basis of Children's Thinking
$8 \times \frac{3}{8} = 8 \times (3 \times \frac{1}{8})$	Relational understanding of $\frac{3}{8}$ as multiple of unit fraction $\frac{1}{8}$
$= 8 \times (\frac{1}{8} \times 3)$	Commutative property of multiplication
$= (8 \times \frac{1}{8}) \times 3$	Associative property of multiplication
$= 1 \times 3$ $= 3$	Relational understanding of $\frac{1}{8}$ and 1 Identity property of multiplication

Figure 4–7. *Justification of a generalizable fraction relationship*

He then said, "I can do that again and then it would be easy to multiply." He continued writing:

$$20 \times \tfrac{3}{4} = 10 \times 1\tfrac{1}{2} = 5 \times 3 = 15$$

Like the strategies above, this one embodied the use of fundamental properties of operations and equality. For example, the first transformation from 20 groups of $\frac{3}{4}$ to 10 groups of $1\frac{1}{2}$ can be justified by the associative property of multiplication. 20 is decomposed into two factors (first line) and then one of the factors is grouped with $\frac{3}{4}$ (second line):

$$\begin{aligned} 20 \times \tfrac{3}{4} &= (10 \times 2) \times \tfrac{3}{4} \\ &= 10 \times (2 \times \tfrac{3}{4}) \\ &= 10 \times 1\tfrac{1}{2} \end{aligned}$$

Possible Equation	Fundamental Property That Is Basis of Children's Thinking
$15 \times \frac{2}{3} = 15 \times (2 \times \frac{1}{3})$	Fraction as a multiple of unit fraction
$= (15 \times 2) \times \frac{1}{3}$	Associative property of multiplication
$= 30 \times \frac{1}{3} = 30 \div 3$	Inverse relationship between multiplication and division

Figure 4–8. *Julie's Relational Thinking strategy*

A Note on Children's Early Relational Thinking Strategies

The strategies presented in this chapter tend to involve the use of fairly efficient relationships. But when children first start using Relational Thinking to solve a particular type of problem, they do not always choose very efficient relationships. Lynne, a fifth grader, used the strategy in Figure 4–9 to solve a problem that asked her to compute $26 \times \frac{3}{8}$. She started with the goal of figuring out how much 10 groups of $\frac{3}{8}$ would be. When we asked her why she chose 10, she said, "When I multiply I like to find out how many 10 groups there are. Then to find out how many 20 groups would be really easy." Ten is often a good number to start with when multiplying a whole number by a whole number, but for this problem, ten was not a very efficient number of groups to start with.

The associative property can also be used to justify transforming 10 groups of $1\frac{1}{2}$ to 5 groups of 3. This strategy is similar to the one described at the beginning of this chapter to solve 5 groups of 12 by using $5 \times 12 = 10 \times 6$.

Children use the fundamental properties of operations and equality naturally in their strategies for problem solving. The use of these properties increases children's understanding of fraction computation at the same time that it increases their understanding of mathematical relationships such as the associative and distributive properties.

As students grow in their use of Relational Thinking, they learn to analyze a problem to determine which relationships would be most efficient in solving that particular problem. Students first need experience deciding for themselves what relationships to use, even if their choices are inefficient, before they can be expected to analyze problems to determine which relationships are best suited to a particular problem. Students refine their anticipatory thinking skills by using relationships to solve problems and hearing how other students used relationships to solve problems.

Relational Thinking Throughout the Mathematics Curriculum

Recognizing and cultivating students' use of Relational Thinking is key to helping them build a deep understanding of fractions and operations on fractions. Profound understanding of arithmetic is marked by the ability to use Relational Thinking to make sense of numbers, operations, and equations. Developing students' Relational Thinking as they are learning fractions integrates their knowledge of whole-number arithmetic with fraction arithmetic, and it lays a critical foundation for future algebra learning.

To appreciate the significance of Relational Thinking across the mathematics curriculum, consider how children might simplify the following expressions and what they have in common (Empson et al. in press):

$70 + 40$

$7/5 + 4/5$

Children who understand place value will see the first expression in terms of combining groups of tens. They would see 70 as 7 groups of 10 and 40 as 4 groups of 10 and figure that together they made 11 groups of 10, or 110. We can represent the "groups of" notion using multiplication:

$$70 + 40 = (7 \times 10) + (4 \times 10) = (7 + 4) \times 10 = 110$$

Children who understand fractions will see the second expression similarly. That is, $7/5$ can be thought of as 7 groups of $1/5$ and $4/5$ as 4 groups of $1/5$. Altogether, there are 11 groups of $1/5$ or $11/5$:

$$7/5 + 4/5 = (7 \times 1/5) + (4 \times 1/5) = (7 + 4) \times 1/5 = 11/5$$

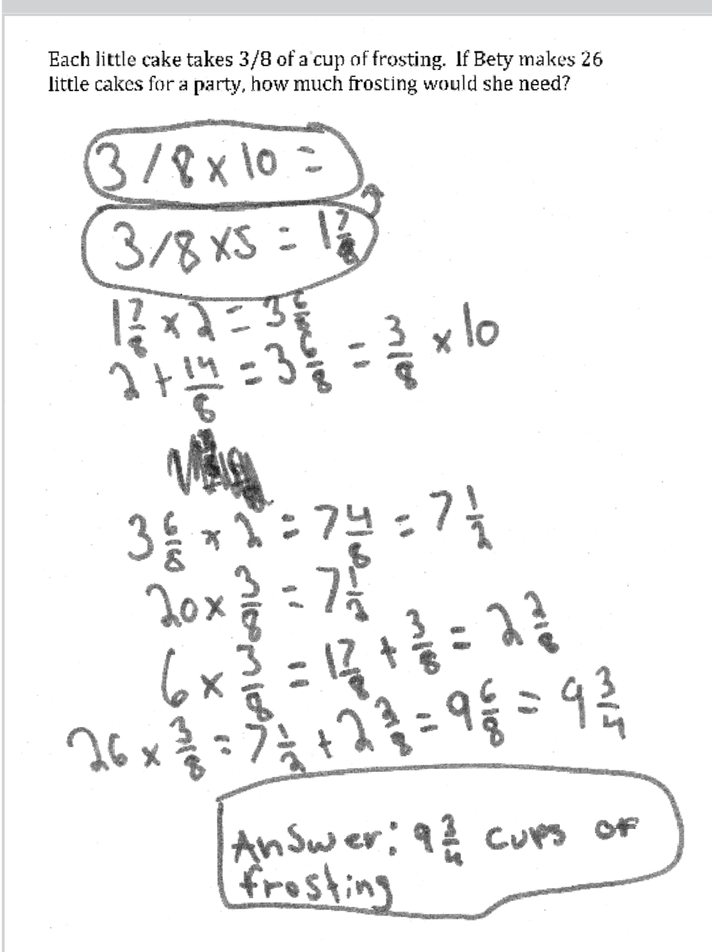

Figure 4–9. Lynne's Relational Thinking Strategy

Each computation is based on the same fundamental property of operations and equality—the distributive property of multiplication over addition. When children work from a well-connected understanding of number and operations, their thinking is based on properties such as this one.

Further, children's understanding of these mathematical properties is the basis for algebra. Consider the justification for simplifying this algebraic expression:

$7a + 4a$

Why does it simplify to $11a$ and not $11aa$ or $11(a + a)$? Like the two computations above, it involves the implicit use of the distributive property:

$$7a + 4a = (7 \times a) + (4 \times a) = (7 + 4)a = 11a$$

$7a$ denotes 7 times a, which can be thought of as 7 groups of size a; similarly, $4a$ can be thought of as 4 groups of size a. Combined, this makes 11 groups of size a, or $11a$. If children learn the addition of whole numbers, fractions, and decimals as a series of rules without also understanding the underlying algebraic basis of the operation—as they so often do—then they will not be prepared to understand algebra and why $7a + 4a = 11a$ rather than $11aa$ or $11(a + a)$. However, if they learn these operations in the context of developing Relational Thinking strategies, they are much more likely to understand the connections between arithmetic and algebra and be able to reason on the basis of mathematical relationships, without a rigid dependence on a fixed set of rules.

The Fundamental Properties of Operations and Equality

Students who use Relational Thinking are using a relatively small set of fundamental properties of operations and equality and related principles to establish connections between quantities, operations on quantities, and equalities between quantities in word problems as well as equations. We have discussed examples of children's use of several of these properties. In Figure 4–10 we provide a list of these properties, which hold for all numbers, including positive and negative whole numbers and fractions, as well as variables representing arbitrary rational numbers. In upcoming chapters, we discuss more instances of children's spontaneous use of these properties in the context of solving problems and performing computations. Children's strategies often make use of more properties than we explicitly identify. You may enjoy the challenge of identifying how children use fundamental properties to guide their thinking beyond the ones that we highlight for a given strategy.

Children's understanding of fractions as relational can be seen as special cases of these properties. For example, understanding that $1 \div n = \frac{1}{n}$ and $n \times \frac{1}{n} = 1$ is a special case of understanding multiplication and division as inverse operations.

“ Teacher Commentary ”

We have found in our district that fractions are often the barrier to understanding algebra. High school teachers don't have time to develop the concepts of fractions. If kids don't learn fractions and really understand them in elementary and middle school, there is little chance to learn fractions in high school. Students often get to algebra and only think of fractions as a picture. They don't see fractions as numbers, and they don't have multiple ways of thinking about fractions. This causes all kinds of difficulties. When you get to solving equations like

$$\frac{x + 2}{2} = 7$$

and you don't understand that fractions are related to division, all you can do is memorize that when you see an equation like this, you multiply each side of the equation by the quantity on the bottom. At some point, memorizing falls apart. Students get to problems that don't match the memorized formulas that they have in their heads, and then they can't solve the problems. Teaching fractions so that students develop Relational Thinking opens the door to higher-level mathematics. Memorizing how to compute does not provide students with the basis for understanding number and operations needed for advanced mathematics. Students who understand fractions are able to make connections, and if they don't remember the rules, they can reason things out and figure out problems. If they get to a problem they haven't seen before, they can often figure out a way to solve it. We are really missing the boat by giving children pictures to color and thinking we are teaching fractions.

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At some point, students need to also learn that addition and multiplication are associative, but division and subtraction are not:

$$24 \div (12 \div 4) \neq (24 \div 12) \div 4$$
$$(92 - 57) - 7 \neq 92 - (57 - 7)$$

And addition and multiplication are commutative, but subtraction and division are not:

$$3 - \frac{1}{2} \neq \frac{1}{2} - 3$$
$$3 \div 2 \neq 2 \div 3$$

Reflecting Back and Looking Ahead

In this chapter, we introduced Relational Thinking and discussed its role in developing students' understanding of fractions. We described how solving and discussing Multiple Groups problems can reinforce concepts of fractions as relational and introduce operations involving fractions. Children develop Relational Thinking strategies as they strive for efficiency in their solutions. When students use Relational Thinking to solve fraction computation problems, they increase their understanding of fraction computation at the same time that they increase their understanding of the fundamental properties of operations and equality. Teaching with a focus on mathematical relationships transforms instruction in fractions into a critical site for the development of algebraic thinking.

Teachers play a necessary role in making the Relational Thinking in children's strategies explicit by writing equations to represent children's thinking and then questioning students about connections between these equations and their thinking. As children develop the ability to use equations to represent their strategies, they gradually learn to reason about equations as objects with mathematical properties—as they need to do when they solve algebraic equations. In Chapter 5, we discuss the teacher's role in eliciting children's Relational Thinking and helping it become an object of reflection. At first, you may simply want to note for yourself how children's strategies make use of Relational Thinking. However, for students to realize the full potential of Relational Thinking, you will need to help them learn to recognize and represent these relationships and use them explicitly in their reasoning.

Properties of Addition		Examples
Identity	$a + 0 = a$	$\frac{3}{8} + 0 = \frac{3}{8}$
Inverse	For every real number a there is a real number $-a$ such that $a + (-a) = 0$	$\frac{1}{3} + (-\frac{1}{3}) = 0$
Commutative	$a + b = b + a$	$\frac{1}{6} + \frac{1}{2} = \frac{1}{2} + \frac{1}{6}$
Associative	$a + (b + c) = (a + b) = c$	$\frac{4}{5} + (\frac{1}{5} + \frac{1}{2}) = (\frac{4}{5} + \frac{1}{5}) + \frac{1}{2}$
Properties of Multiplication		**Examples**
Identity	$a \times 1 = a$	$\frac{4}{3} \times 1 = \frac{4}{3}$
Inverse	For every real number a, $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$	$8 \times \frac{1}{8} = 1$
Commutative	$a \times b = b \times a$	$9 \times \frac{2}{3} = \frac{2}{3} \times 9$
Associative	$a \times (b \times c) = (a \times b) \times c$	$(5 \times 4) \times \frac{3}{4} = 5 \times (4 \times \frac{3}{4})$
Distributive Property of Multiplication over Addition		**Example**
	$a \times (b + c) = (a \times b) + (a \times c)$	$6 \times 2\frac{1}{3} = 6 \times (2 + \frac{1}{3}) = (6 \times 2) + 6 \times \frac{1}{3})$
Other Properties of Operations		**Examples**
Addition and Subtraction are Inverse Operations	If $a + b = c$, then $c - b = a$	$\frac{3}{4} + \frac{1}{4} = 1$, so $1 - \frac{1}{4} = \frac{3}{4}$
Multiplication and Division are Inverse Operations	If $a \times b = c$, then $c \div b = a$	$8 \times \frac{3}{4} = 6$, so $6 \div \frac{3}{4} = 8$
Properties of Equality		**Examples**
Addition Property of Equality*	If $a = c$, then $a + b = c + b$	$\frac{1}{3} = \frac{1}{6} + \frac{1}{6}$, so $\frac{1}{3} + \frac{1}{6} = (\frac{1}{6} + \frac{1}{6}) + \frac{1}{6}$
Multiplication Property of Quality*	If $a = c$, then $a \times b = c \times b$	$3 \times \frac{2}{3} = 2$, so $5 \times (3 \times \frac{2}{3}) = 5 \times 2$

*We do not include a subtraction property of equality or a division property of equality, even though they are true properties, because subtraction can be expressed in terms of addition and division in terms of multiplication.

Figure 4–10. *Fundamental properties of operations and equality*