Invigorating HIGH SCHOOL Math

Practical Guidance for Long-Overdue Transformation

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The authors and publisher wish to thank those who have generously given permission to reprint borrowed material:

Figure 1.2, Figure A.1, Figure A.2: "Figure 2: Long Term (2023–24) Course Pathway Options" and "Table 3: Total count of core high school standards in January 2021 draft" from *2021 Oregon Draft Mathematics Standards: High School Core Mathematics*, January 2021 Draft for Public Review. Reprinted by permission of the Oregon Department of Education.

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Library of Congress Control Number: 2021910414
ISBN: 978-0-325-13416-1

Editor: Heather Anderson
Production: Victoria Merecki
Cover design: Vita Lane
Cover images (l to r): © SDI Productions/Getty Images; © Prostock-studio/Adobe Stock; © SDI Productions/Getty Images
Interior design and typesetting: Gina Poirier, Gina Poirier Design
Manufacturing: Val Cooper

Printed in the United States of America on acid-free paper
1 2 3 4 5 VP 25 24 23 22 21
July 2021 Printing

Contents

CHAPTER

3

Change is the law of life, and those who look only to the past and present are certain to miss the future.

—JOHN F. KENNEDY (1963)

Guiding Principles

Domains of Invigoration

THIS CHAPTER INTRODUCES fourteen guiding principles or domains of invigoration that require serious consideration, discussion, planning, and, when appropriate, implementation. These domains, summarized in Figure 3.1, arise from the challenges we have already delineated and from our experiences in high schools and with mathematics teachers across the country.

Goals and Purposes

GUIDING PRINCIPLE

The high school mathematics program must have a shared, written, and honored set of goals and purposes that guide decision-making, policy, and program. These living principles must address the unique role of mathematics in our society and the critical need for mathematical literacy on the part of all students. They must include such purposes as understanding and critiquing the world; experiencing wonder, joy, and beauty; and expanding professional opportunity, instead of merely preparing students for the next course and for required college entrance examinations.

DOMAINS OF INVIGORATION FOR HIGH SCHOOL MATHEMATICS

DOMAIN/ PRINCIPLE	SUMMARY	CHAPTERS WHERE ELABORATED
GOALS AND PURPOSES	The high school mathematics program must have a shared, written, and honored set of goals and purposes that guide decision-making, policy, and program.	Chapter 3
VISION	The high school mathematics program must be guided by a descriptive and explicit vision of effective teaching and learning of mathematics.	Chapter 3
EQUITY AND ACCESS	The high school mathematics program must ensure that all students have access to a high-quality mathematics curriculum and high-quality instruction.	Chapter 6
CULTURE	The high school mathematics program must operate within a culture of mutual respect, honoring the dignity of all, and collegial collaboration between and among the adults and the students.	Chapter 6
SCOPE	The high school mathematics program must focus on the essential ideas and processes of mathematics.	Chapters 4 and 5
DIFFERENTIATION	The high school mathematics program must be built around a common core of essential and important mathematics differentiated by life and career goals.	Chapters 5 and 6
INTEGRATED MATHEMATICS	The high school mathematics program must recognize that mathematics is a unified body of knowledge about quantity, change, uncertainty, shape, and dimension.	Chapters 4 and 5
CONTENT AND PROCESS STANDARDS	The high school mathematics program must be driven by standards or expectations that delineate the learning goals for each course and each unit within each course.	Chapters 4, 5, and 6
CONNECTIONS	The high school mathematics program must include explicit and coherent exploration of how one piece of mathematics relates to other pieces of mathematics.	Chapters 4, 5, and 6

FIGURE 3.1

continues

DOMAINS OF INVIGORATION FOR HIGH SCHOOL MATHEMATICS, *cont.*

DOMAIN/ PRINCIPLE	SUMMARY	CHAPTERS WHERE ELABORATED
CONTEXT AND MODELING	The high school mathematics program must include situations, applications, and contemporary problems, often interdisciplinary in nature, that illustrate the usefulness of mathematics.	Chapter 9
ASSESSMENT	The high school mathematics program must recognize assessment as an integral part of instruction.	Chapter 7
TECHNOLOGY	The high school mathematics program must make full use of technologies that increase the productivity of instruction and enrich students' experiences.	Chapter 8
ADEQUATE TIME	The high school mathematics program must have adequate time to effectively meet the learning standards and implement the vision of teaching and learning that guides our work.	Chapter 10
PROFESSIONAL GROWTH AND COLLABORATION	The high school mathematics program must be supported by intensive attention to ongoing professional growth and collaboration among the mathematics teachers.	Chapter 10

FIGURE 3.1, *continued*

Without clarity of goals and purposes, we meander, we invite incoherence and conflict, and we underserve. Consider the importance of a clear, stated learning goal for each lesson. With such a goal in mind, we are better able to identify prerequisite knowledge, select appropriate tasks, ask the right questions, and construct the right formative assessment exit ticket. Conversely, without pausing to clarify a lesson goal, too often we end up punting—showing, telling, and providing opportunities to practice instead of teaching for depth and broader purposes.

The same holds true for a mathematics department. We believe that change begins with discussions that address the questions "What are the specific goals of four years of mathematics courses?" and "What are the purposes of learning mathematics in our high school?"

In *Catalyzing Change in High School Mathematics*, the National Council of Teachers of Mathematics (NCTM) urges us to consider a three-part purpose of school mathematics that includes expanding professional opportunity, understanding and critiquing the world, and experiencing wonder, joy, and beauty (2018, 9–12). The implications of this shift in purpose are significant. To expand professional opportunity, high school mathematics must fully consider the diverse future plans and accompanying mathematical needs of students. For students to better understand and critique the world, high school mathematics must ensure that mathematical experiences get connected to situations, problems, and phenomena of interest. And to ensure that students experience wonder, joy, and beauty in lieu of drudgery and pain, high school mathematics must be infused with tasks and questions and projects that bring out that wonder and beauty. One of our goals throughout this book is to describe and exemplify these broader purposes of high school mathematics.

Here is a sample statement of goals and purposes that we hope can be used to start discussions, make revisions, and eventually build consensus on the development of a goal statement that you and your colleagues can use and live by:

The overarching purpose of the High School Mathematics Department is to mathematically empower every student over the course of four years of relevant, applicable, and appropriate mathematics coursework. Our efforts to achieve this ambitious purpose, as advocated by the National Council of Teachers of Mathematics, are guided by our desire to provide all students with the opportunity to develop competence in using mathematics to understand and critique the world and its social and natural phenomena; ensure that all students experience wonder, joy, and beauty when learning mathematics; and expand all students' career opportunities after high school.

Accordingly, the overarching goals of the High School Mathematics Department are to

- provide a meaningful and engaging program of studies, including a common integrated essential core followed by differentiated pathways to prepare all students for college and the world of work
- implement instructional practices and department policies that provide equitable access to important mathematics for all students
- ensure that in every class students are expected to ask penetrating questions, explain their thinking, make reasonable estimates and predictions, and justify and respond to one another's mathematical arguments, strategies, and decisions
- ensure that all mathematical skills and concepts are taught within a context of reasoning, problem solving, and mathematical modeling.

This statement provides a broad, clear statement of purpose that explains why we are doing what we do and a short set of goals that define, in specific terms, how we plan to accomplish this purpose. We envision such a revised and broadened statement of goals and purposes being front and center on the department's web page, in the mathematics department section of the course catalog, and in poster form in every classroom where mathematics is taught.

In another school, the mathematics department was challenged to create a definition of a world-class mathematics program as an aspirational statement with which to periodically assess the overall quality of K–12 mathematics. Here is what the teachers drafted:

World-Class Mathematics at Jefferson High School: A Shared, Written Definition of Excellence

As it pertains to Jefferson High School, our mathematics program

- blends systematically the development of a strong foundation of mathematical skills, conceptual understanding and applications, and the valuing of academic success, found in the highest-performing East Asian countries, with the focus on problem solving, investigation, reasoning, justification, and modeling, found in the most successful Western approaches, to create an empowering intercultural mathematics program
- provides coherent alignment of a shared vision, progressions of learning objectives, curriculum materials, instructional practices and assessments, and differentiated approaches of teaching strategies and student learning support—with each of these components thoughtfully implemented and revised based on accepted research findings and student performance data
- ensures that all students experience the beauty, awe, and joy of mathematics; develop curiosity, perseverance, self-confidence, and the ability to argue and justify mathematically; and build number sense, symbol sense, spatial sense, and data sense throughout their mathematical experiences, as parts of a mathematics tool kit that gets expanded and refined each year
- pledges, through collaborative structures and intentional professional growth opportunities, that all teachers will have the time, technological tools, resources, and support to effectively implement the program with high levels of effectiveness

- builds a schoolwide culture, including students, teachers, administrators, and parents, that values mathematics, sees the value of a world-class mathematics program that addresses the needs and demands of our ever-changing world, and supports this shared vision of a world-class program
- confirms, through differentiated approaches to teaching and learning in every mathematics classroom, and targeted resources, that high levels of mathematical skill and understanding are achieved by each and every student.

It is easy to draft fancy and flowery language. There is a long history of ignoring such words and filing away these lofty statements of goals and purposes. Our hope is that these two sample statements serve as initial thought provokers to help stimulate discussion and drafting of statements that work for you and your colleagues.

Questions to Ask Yourselves

- Where in your stated goals or purpose statement is there reference to effective citizenship or to the demands of preparation for a changing world?
- Where in your stated goals or mission are there references to the fundamental purposes of learning mathematics?
- Where in your stated goals or mission is there reference to common essential content for all and differentiated pathways to meet different individual aspirations?

Vision

GUIDING PRINCIPLE

The high school mathematics program must be guided by a descriptive and explicit vision of effective teaching and learning of mathematics—a statement that conveys classroom norms and expectations for professional behavior on the part of students, teachers, and administrators. Effective vision statements support the implementation of the goals and purposes of the program through consistent use of research-affirmed pedagogical practices.

While a statement of goals and purposes frames a large view of what we do and serves as a message to all stakeholders, a teaching and learning vision statement is where we share among ourselves exactly what we should and should not be seeing in classrooms where mathematics is being taught. It defines expectations for professional behavior. In an invigorated high school mathematics department, a shared vision statement—when widely bought into and used—guides pedagogical decision-making, coaching and supervision support, and even teacher evaluation.

We draw from NCTM's eight mathematics teaching practices as defined and described in *Principles to Actions* (2014) to help us craft a vision of what these practices look like in mathematics classrooms. Here is one straightforward vision statement for consideration:

Our vision of effective teaching and learning of mathematics is based on our understanding of what common sense, the wisdom of practice, and research tell us about maximizing learning in our classroom. At our high school, when mathematics is being taught, planning will ensure, implementation will establish, and observation will confirm the following:

- clear statements and brief discussion of lesson goals, instead of "Today we are doing Lesson 4.5 or pages 214–217"
- use of context whenever appropriate, instead of just naked numbers
- use of rich tasks, instead of just worksheets of exercises
- focused oral *and* written questions, instead of just making things up as we go along
- frequent opportunities for, and expectations of, student discourse, instead of just telling
- gradual reveal of tasks, problems, and solutions, instead of just dumping whole paragraphs
- consistent use of multiple representations, instead of just numeric or symbolic representations
- consistent seeking and presenting of alternative approaches, instead of just providing one way to get an answer
- clear expectations for explanations and justifications, instead of just focusing on correct answers
- valuing common errors and misconceptions, instead of just a search for right answers

- consistent expectations that the mathematics will make sense as students construct understanding, instead of just lecturing, showing, and practicing
- gathering and reviewing evidence of learning or the lack thereof, instead of just "I taught it and let the chips fall where they may."

Here is an alternative model adapted from a network of charter schools:

A Guiding Vision for Inquiry-Based, Conceptually Driven, Sense-Making Mathematics in Every Mathematics Class Every Day

If our shared commitment is that every Washington High School student receives well-planned, well-executed mathematics instruction that consistently reflects our vision of active engagement in thought-provoking tasks, productive discussion about mathematical ideas and common misconceptions, and the individual and collective construction of understanding via problem solving and inquiry,

THEN this commitment requires that teachers plan their lessons around **rich tasks** that are supported by **targeted questions** and powerful **lesson debrief discussions**. Such lessons are diametrically opposite to the "I show, we practice, you do" model of direct instruction that essentially tells students what to remember and how to get right answers. For example, the "trick" to "invert and multiply" (as opposed to understanding that dividing by a number is the same as multiplying by the reciprocal of that number) works in the short term, but does not support mathematics as a sense-making enterprise and does not foster an inherent love of mathematics and its power and beauty.

MOREOVER, the problem we face as a community of teachers, administrators, and parents is that our vision is **not** widely shared, **not** fully understood or even believed, **not** consistently supported, and therefore **not** consistently implemented for all students every day. To begin to address this problem, the chart below [Figure 3.2] summarizes what students, teachers, and leaders are, and are not, doing to make inquiry-based, conceptually driven, sense-making mathematics the norm in every Washington High School mathematics class.

WHAT **STUDENTS** ***ARE*** DOING	WHAT **TEACHERS** ***ARE*** DOING	WHAT **LEADERS** ***ARE*** DOING
▪ Actively engaging in solving rich problems that are aligned with the curriculum standards ▪ Regularly engaging in productive discourse about their thinking and reasoning ▪ Grappling with mathematical ideas and making and exploring conjectures about those mathematical ideas	▪ Thoroughly studying the curriculum standards, the textbook, and other resources to develop an understanding of the key mathematical understandings across a grade, unit, or lesson ▪ Carefully selecting rich tasks that support reasoning and problem solving ▪ Anticipating students' solutions and strategies for each task ▪ Carefully crafting and asking targeted questions that focus on the key mathematical understandings ▪ Making frequent use of the critical questions "Why?" "Can you explain?" "Who did it differently?" "Can you convince the class?" and "How did you picture that?" ▪ Regularly collecting and using formal and informal evidence to assess scholar understanding of the big mathematical ideas and adjusting their instruction accordingly	▪ Regularly meeting with teachers to help them think through their lesson plans, including clarifying the learning goal and selecting rich, aligned tasks and the questions to be asked during the lesson ▪ Co-teaching the lesson in ways that support the teacher and maintain a focus on the learning goals ▪ Taking notes to support a productive debriefing and action-planning session

WHAT **STUDENTS** ***ARE NOT*** DOING	WHAT **TEACHERS** ***ARE NOT*** DOING	WHAT **LEADERS** ***ARE NOT*** DOING
▪ Solving more than three naked problems from a worksheet without the chance to explain their thinking ▪ Listening to explanations by the teacher without interruption ▪ Regurgitating procedures to get answers	▪ Showing students how to solve problems and expecting them to replicate the process solely on the basis of remembering ▪ Using the phrase "this is the rule" or "this is how you solve this" or "this is what you have to remember" without including reasons, explanations, or a focus on why ▪ Allowing students to solve problems without providing any opportunities for feedback	▪ Sitting on the sidelines, without interrupting or participating in the lesson ▪ Using the co-teaching and coaching process only for evaluation ▪ Using only co-teaching and coaching, without providing opportunities for preplanning and debriefing

FIGURE 3.2

In creating these statements, like all great teachers, we borrow and steal from others for initial drafts and then collaboratively and iteratively craft statements upon which the entire department can agree. Often the process is as important as the final product as we grapple with building consensus around our professional expectations for ourselves and our colleagues.

Questions to Ask Yourselves

- Does your department have a vision statement that describes what should be happening in effective mathematics classrooms? If not, why?
- What might be the benefits of collaboratively developing such a statement?
- What's to stop you from starting with the above examples and collaboratively creating a vision statement for your high school mathematics department that works for you and your colleagues?

Equity and Access

GUIDING PRINCIPLE

The high school mathematics program must ensure that all students have access to a high-quality mathematics curriculum and high-quality instruction. The program sets high, but reasonable, expectations and provides the support and resources needed to ensure that these expectations are attainable by all students.

It is a sad but all too common reality that school mathematics, particularly high school mathematics, acts as a filter, limiting access and denying opportunities for many students, often with the best of motives. When some students get the best teachers, have consistent access to technology, and embrace a curriculum that opens doors, while two doors down the hall or in the school down the street, other students get the least experienced teachers, have a primarily pencil-and-paper skill-based program, and endure a curriculum that is essentially a dead end, there is a serious problem.

In too many cases and places, high school mathematics continues to be a powerfully effective sorting machine, ensuring that far too many students leave high school woefully unprepared for the world of postsecondary education and work despite having taken three or four years of high school mathematics. In our experiences, in too many schools and districts this situation emerges from a conspiracy of well-meaning teachers, misguided counselors, and, too often, upper-middle-class parents concerned only about maintaining the relative privilege of their children.

To rectify this severely inequitable situation, we turn to NCTM's *Principles to Actions* (2014) and focus on the core "productive belief" that "mathematics ability is a function of opportunity, experience and effort—not of innate intelligence. Mathematics teaching and learning cultivate mathematics abilities. All students are capable of participating and achieving in mathematics, and all deserve support to achieve at the highest levels" (63).

How different this "productive belief" is from the far more common and insidious unproductive belief that "students possess different innate levels of ability in mathematics, and these cannot be changed by instruction. Certain groups or individuals have it while others do not" (63). Until beliefs like this are banished from our profession, high school mathematics will never be invigorating or designed in ways that truly serve all students.

But what specific actions ensure "opportunity, experience and effort?" We begin by advocating for the equitable teaching practices enumerated in NCTM's *Catalyzing Change in High School Mathematics* (2018, 32–34) and discussed in greater detail in Chapter 6. Access and equity begin with our instructional practices and the array of behaviors that enable all students to feel socially, emotionally, and intellectually safe and valued in every classroom. Too often we have hidden behind the excuse that we teach *mathematical content* to students and, if it doesn't work, the problem is most often the student. We need to recognize that our job is to teach *students* mathematics and that their feelings of safety, belonging, and being valued as people and learners ultimately determine how much they will learn and succeed.

In addition, we implement the vision of this guiding principle—providing opportunity and experiences—when we have a system of retesting for every unit assessment to provide second chances (more in Chapter 7), and when we replace the pernicious system of tracking with a much fairer system of selective acceleration and differentiated enrichment, as described in Chapters 4 and 5.

Questions to Ask Yourselves

- In what ways does your high school mathematics department provide access and equity to the students in your school?
- In what ways, when you are honest with yourselves, do you recognize that your high school mathematics department fails to maximize access and equitable opportunity to all students?

- How can you justify a system of tracks in high school mathematics that anoints some for a future of success and relegates others to a future with very limited opportunities?
- What specific shifts in practice and policy do you believe will significantly enhance access and equitable opportunity for the students in your school?

Culture

GUIDING PRINCIPLE

The high school mathematics program must operate within a culture of mutual respect, honoring the dignity of all, and collegial collaboration between and among the adults and the students in which the empowerment of both teachers and students drives what we do and how we do it. Even when goals and purpose statements, along with vision statements and policies to ensure equity and access, are in existence, they are only as meaningful as their implementation in the form of a shared culture that pervades how the mathematics department operates.

It is a truism of all organizations that "culture eats strategy for breakfast"—a phrase attributed to management guru Peter Drucker. However, few high school mathematics departments spend much time thinking about the culture in which they operate and which they wittingly and unwittingly communicate through actions every minute of every day.

In this case, adapting a definition from online dictionaries is helpful because mathematics department culture is generally not a topic of much discussion:

> A culture consists of the shared customary beliefs, social norms, and typical behaviors or habits of a particular social group. Culture is the set of shared attitudes, values, goals, and practices that characterizes an organization or group.
>
> Humans acquire culture through the processes of enculturation and socialization in which cultural norms codify acceptable conduct and guide behavior, language, and demeanor—thereby serving as a template for expectations within a social group.

In an invigorated, effective, and equitable mathematics department and program, there is a tangible sense of *belonging* and being *valued*. These characteristics define the culture. Students are never put down or made to feel dumb. Instead, they are welcomed into a mathematics learning community, their thinking is celebrated, their mistakes are seen as learning opportunities, and they know they are in an intellectually and emotionally safe place. In short, regardless of their ability, the mathematics classroom is felt to be a place where they belong and can thrive.

And it is easy to see and feel the ambient culture in classrooms. Just imagine randomly walking into three or four mathematics classrooms in your high school on any given morning. Are the desks all in rows in every room, or are students, and their desks or tables, organized in pairs, trios, or quartets, or is it some of each from room to room? How about what the students are doing? Are they actively engaged, working together, and using technology to solve what appear to be engaging tasks, or are they sitting compliantly listening to the teacher lecture, or do you see some of each? What's on the walls? Is the lesson goal clear and posted? Is an interactive whiteboard being used, or do you see effective use of animated slides on a screen, or is everything still being done with markers on a whiteboard by the teacher in the front of the room? Do you see similar approaches and problems in each algebra or geometry class, or does it look like students with two different teachers are in two different schools? How is homework reviewed? Does it count toward a grade? Is "going over homework" just a recitation of right answers? Are there ever other teachers or other adults in any of the classes you observe? What are they doing to support teaching and learning? And what about how students are treated? Is there sarcasm? Is there humor, and are there smiles? Consider the answers to these and related questions and you get a fairly powerful sense of the degree to which the culture is conducive to learning, caring, and working hard—or not.

Then sit in on your department's monthly meeting. Is there a written, shared agenda? How much time is allocated to schedules and budgets and textbook orders and the latest dictates from the administration? How much time is allocated to discussions of "tricky and troublesome topics" or "an amazing app" that one colleague models for everyone else or a ten-minute video of a colleague's lesson the day before that is presented for analysis and critique? Obviously, some of these agenda items have nothing to do with the quality of teaching and learning or the development of a collaborative culture of improvement and mutual support among the members of the department. And just as obviously, other agenda items characterize a department as a team of professionals forever striving to build a culture of quality, mutual respect, and ongoing improvement.

The answers to these and other questions about classroom practice and department meetings provide a clear sense of the culture you establish and within which you work. They tell us about the norms and behaviors and expectations. They tell us whether there is a common shared culture or an "I'm okay, you're okay" laissez-faire approach. But regardless of the specific answers to such questions, it should be clear that culture dominates. To invigorate high school mathematics requires examining and adjusting culture.

Questions to Ask Yourselves

- How would you describe the dominant aspects of the culture within your classroom?
- How does your description align with or differ from descriptions provided by other members of the department?
- How would you describe the collective culture of your mathematics department?
- What suggestions do you have for enhancing this culture and creating a healthier and more productive environment?

Scope

GUIDING PRINCIPLE

The high school mathematics program must focus on the essential ideas and processes of mathematics. There must be an emphasis on the development of understanding and application of important content, rather than an effort to teach too much too quickly and with too little depth. Mathematics topics may be considered important for different reasons, such as their usefulness in building foundations for developing other mathematical ideas, their value for representing and solving problems, their role in linking mathematics to other disciplines, and their ability to deepen students' appreciation of mathematics as a discipline and a human endeavor.

If we are to truly invigorate high school mathematics, it is essential that we acknowledge that it is essentially impossible to complete an entire textbook in any given high school mathematics course unless we race through the text at an absurd pace with little depth and few opportunities for application and modeling. That's why effective teachers, and wisely developed curricula and pacing guides, use the textbook only as a resource and not a bible, identifying which chapters and which lessons must be skipped in order to make the course teachable. Similarly, it is essential that we acknowledge that most high school courses—particularly those outlined in the pernicious Appendix A of the Common Core—are, with almost sixty standards per course, essentially unteachable. We—effective teachers and knowledgeable curriculum developers—must use our experience to design truly teachable courses that employ technological tools, minimize obsolete skills, and focus on depth of understanding, problem solving, and modeling, leaving ample time for activities, tasks, projects, and discussions that support learning.

That is why, as we argued in Chapter 1 and suggest later in Chapters 4 and 5, our course proposals do not include such topics as factoring binomials with leading coefficients not equal to 1, division of polynomials and synthetic division, trigonometric identities, memorizing theorems (as opposed to selecting, using, and proving them), rationalizing denominators, and using pencil and paper to solve equations that technology can solve far more efficiently. We advocate for courses with a maximum of six units a year, with approximately thirty days per unit, so that there is adequate time to explore, teach, and learn mathematics in depth.

Questions to Ask Yourselves

- For which courses in your mathematics department do you feel there is an unreasonable amount of mathematics to "cover"? Why does this situation exist?
- Which of the topics that you currently teach have questionable value for students?
- Can you try to organize each existing high school mathematics course into five or six approximately thirty-day units? What must be eliminated in order to accomplish this?

Differentiation

GUIDING PRINCIPLE

The high school mathematics program must be built around a common core of essential and important mathematics differentiated by life and career goals. It must be delivered through a balance of shared experiences for all students and learning tasks that are appropriately chosen to reflect the prior knowledge of the students and respond to and build on that knowledge. Differentiation requires creating, modeling, adapting, and enriching instruction so that it engages the student, corrects misconceptions, sustains interest, and promotes confidence and perseverance.

We start with the simple reality that there are plus or minus twenty-five distinct brains in every class of students. We add the obvious truism that "one size never fits, doesn't fit, and can't ever fit all." And we conclude that we can only maximize the learning of mathematics by differentiating.

To people who have never taught or who choose to ignore the realities of socially mediated learning, differentiation is as simple as placing every student on a personalized computer with a personalized program. That is *not* how we perceive differentiation. Rather, recognizing both the differences among any class of students and the need for interaction and collaborative learning opportunities, we argue that differentiation is as simple and as challenging as a continuous focus on alternative approaches and multiple representations. Given that the one right way to get the one right answer has never served all students, effective teachers have always differentiated by asking "Who did it differently?" to elicit alternative approaches. And given that limiting instruction to only symbolic representations of mathematics has also never served all students, effective teachers have always differentiated by asking "How did you see that?" and "Who saw that differently?" to elicit a broader range of representations. We have touched upon this critical instructional shift in the discussion of vision earlier in this chapter, and we return to it in Chapter 6, on issues of pedagogy.

But differentiation is not just a classroom practice—it must also be built into a set of pathways that honor our students' different mathematical needs and future aspirations. We describe such a set of differentiated pathways in Chapter 5.

Questions to Ask Yourselves

- In what ways does our department's standard practice reflect a one-size-fits-all approach?
- In what ways does our department differentiate to maximize the opportunity for all students to learn?

Integrated Mathematics

GUIDING PRINCIPLE

The high school mathematics program must recognize that mathematics is a unified body of knowledge about quantity, change, uncertainty, shape, and dimension—or more formally, number, variables and functions, probability and data, and geometry and measurement. We have packaged this knowledge into algebra, geometry, and statistics silos, ignoring the connections between these silos and the power of making links between and among the different strands of the mathematical sciences.

One of the great mysteries of high school mathematics in the United States is how an integrated K–8 curriculum that balances number, algebra, data, shape, and measurement every year culminates in a high school program that ignores integration and resorts to Algebra 1, Geometry, Algebra 2, and Precalculus silos as the privileged pathway. No other industrialized, first-world country in the world follows this approach; instead, they use an integrated curriculum through grade 12. While the United States places AP Calculus on the pinnacle of achievement, much of the rest of the world, including many countries that far outperform the United States, considers the integrated International Baccalaureate (IB) program the gold standard for achievement.

Our strong bias toward an integrated program stems from

- the difficulty of inserting enough statistics and data analysis into the traditional program
- the reality that geometry would be well served with two-thirds of the time it now has
- the critical need to significantly increase the connections between and among the different domains of mathematics.

With more algebra taught in eighth grade than ever before and statistical literacy increasingly recognized as a critical component of mathematical understanding, an integrated approach makes even more sense. We suggest how this integration can be accomplished in Chapters 4 and 5, where we propose a common, integrated mathematics sequence of courses in grades 9 and 10, followed by differentiated pathways of courses in grades 11 and 12 that embrace appropriate connections between algebra, geometry, statistics, and topics in discrete mathematics.

Questions to Ask Yourselves

- Why do we cling to the traditional sequence as the core of high school mathematics for the vast majority of students?
- What are some of the distinct advantages *and* disadvantages of moving to an integrated approach, particularly in grades 9 and 10?
- What is the decision-making process in your school or district that could lead to a shift from the traditional sequence to an integrated approach?

Content and Process Standards

GUIDING PRINCIPLE

The high school mathematics program must be driven by standards or expectations that delineate the learning goals for each course and each unit within each course. Standards—both those that describe content and those that focus on process or practice—are the skeleton of a program and delineate specifically what students should know and be able to do as a result of completing a unit or course. These standards or learning goals guide decisions about how to use a textbook, what supplemental materials are necessary, and the content of all assessments.

We have come a very long way in moving from textbook-driven mathematics courses to standards-driven mathematics courses. Way back in 1989, NCTM's *Curriculum and Evaluation Standards for School Mathematics* began the national movement toward more precise descriptions of the expectations for student learning and introduced the radical notion of process standards to accompany content standards. It took two decades of meandering around in what proved to be valuable, but very general, grade-band statements of expectations to get to the Common Core State Standards for Mathematics in 2010 that, in terms of poker, "saw" NCTM and "raised" it with much greater specificity and guidance in the realms of both content and process/practice.

As we have noted, however, this specificity and guidance from the Common Core fell far short for grades 9–12, and many high school mathematics departments continue to wrestle with such critical questions as

- What specific content must be addressed in which course?
- What scope or number of standards is actually teachable, even with an hour a day?
- What progressions or trajectories of skills and concepts (a key feature of the K–8 standards) apply to 9–12?
- What balance should we strike between content and practice, and how are the critical practice standards interwoven into teaching the content?

Our curricula and our course guides must answer these questions. At their best they delineate content standards and describe, with examples, specifically what needs to be taught and learned, as well as the practices that transcend specific content and guide instruction.

When it comes to the more elusive process or practice standards, we are reminded of the progression from NCTM 1989 to NCTM 2000 to the Common Core in 2010, as shown in Figure 3.3. Moreover, even though most states have altered the original Common Core mathematics standards, every one of them has retained the mathematical practices as a key component of their state standards.

EVOLUTION OF PROCESS/PRACTICE STANDARDS		
NCTM 1989 CURRICULUM AND EVALUATION STANDARDS	**NCTM 2000 PRINCIPLES AND STANDARDS**	**2010 COMMON CORE STANDARDS FOR MATHEMATICAL PRACTICE**
1. Problem solving 2. Reasoning 3. Communication 4. Connections	1. Problem solving 2. Reasoning 3. Communication 4. Connections 5. Representation	1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.

FIGURE 3.3

We note, with respect to these practice/process standards, how much we all struggled to incorporate the original four 1989 process standards and how much confusion reigned on how and where to best address these critical elements of mathematics. We note that in 2000, NCTM recognized a giant hole and added representation, suggesting that these processes, like everything we do, need to be rethought and adapted over time. But once again, we struggled to address what were now five critical elements of mathematics. And then came the shift from "processes" to "practices" and the increase to *eight* practices proposed in the Common Core. If teachers were understandably overwhelmed with just four or five, how could anyone expect eight to take root in our classrooms?

Our advice is to limit expectations to what has a reasonable chance of being implemented, and to take seriously the risks of overwhelming professionals with too many expectations. Our advice is to make the first four Common Core Standards for Mathematical Practice the nonnegotiable overarching focus in everything we do in K–12 mathematics:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

Our purpose is *not* to belittle precision, tools, structure, and regularity, but to keep the focus on what matters most: problem solving, reasoning, argument, and modeling. Because there is so much that needs to be done to invigorate high school mathematics, when we can limit the demands, we increase the probability of effective implementation and success. Most importantly, we believe that these four statements capture exactly those practices that align with any set of updated goals or purposes for high school mathematics.

Just think about a society where every high school graduate could make sense of unfamiliar problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and comfortably critique the reasoning of others, and use mathematical understanding to model real-world phenomena. What a world that would be, and what a wonderful goal to which every teacher of high school mathematics might aspire!

Questions to Ask Yourselves

- How successful have you and your colleagues been in implementing the vision of the eight Common Core Standards for Mathematical Practice?
- Have you and your colleagues had discussions and arguments about whether all eight practices are equally important? If not, why?
- If you start with our proposed four practices, how might you and your colleagues revise and adjust these practices for implementation in your department?

Connections

GUIDING PRINCIPLE

The high school mathematics program must include explicit and coherent exploration of how one piece of mathematics relates to other pieces of mathematics. It must effectively organize and integrate important mathematical ideas so that students grasp how the concepts, skills, and logical thinking build on and connect to each other.

A core, but often ignored, tenet of educational psychology is that we learn by connecting new knowledge to previously learned knowledge and that we develop new understandings by building on connections to existing understandings. Without connections, each bit of knowledge sits alone, making it much harder to process and more difficult to apply. Effective teachers have forever asked, "How is this like what we have already studied?" and "How is this similar to and different from this previous body of knowledge?" They are fostering connections and strengthening the mental power of learners of all ages.

Back in 1989, NCTM's *Curriculum and Evaluation Standards for School Mathematics* stated for Standard 4: Mathematical Connections in grades 9–12 that

> the mathematics curriculum should include investigations of the connections and interplay among various mathematical topics and their applications so that students can:
>
> - recognize equivalent representations of the same concept;
> - relate procedures in one representation to procedures in an equivalent representation;
> - use and value the connections among mathematical topics;
> - use and value the connections between mathematics and other disciplines. (146)

For the same standard eleven years later, in NCTM's updated *Principles and Standards for School Mathematics* (2000), we find:

> When students can see the connections across different mathematical content areas, they develop a view of mathematics as an integrated whole. As they build on their previous mathematical understandings while learning new concepts, students become increasingly aware of the connections among various mathematical topics. As students' knowledge of mathematics, their ability to use a wide range of mathematical representations, and their access to sophisticated technology and software increase, the connections they make with other academic disciplines, especially the sciences and social sciences, give them greater mathematical power. (354)

It is hard to imagine teaching the distributive property as only a symbolic rule that $a(b + c) = ab + ac$ without also showing and calculating the areas of a rectangle with length $b + c$ and width a that concretely models the area of the entire rectangle ($a(b + c)$) and the area of the two parts ($ab + ac$). Similarly, teaching completing the square without pausing and completing a physical square with algebra tiles ignores the power of connections and borders on malpractice (see Figure 3.4).

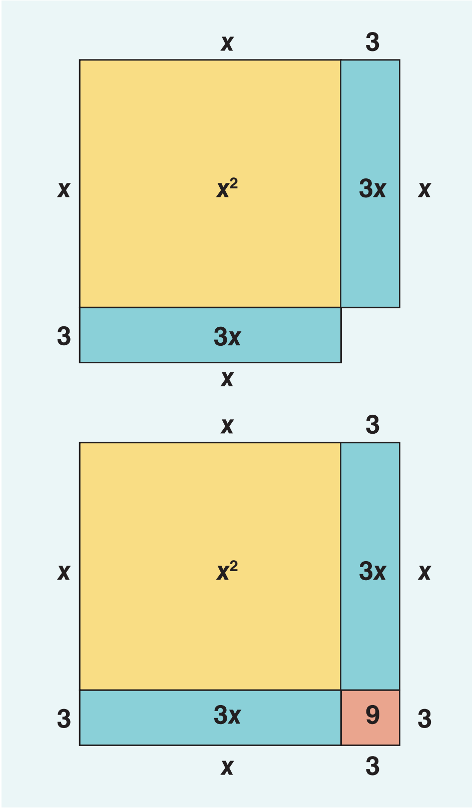

FIGURE 3.4

Questions to Ask Yourselves

- In a recent unit you taught, what opportunities did you use to make connections between mathematical topics, among equivalent representations, and between mathematics and other disciplines?
- In the next unit you will be teaching, what opportunities do you foresee for making connections between mathematical topics, among equivalent representations, and between mathematics and other disciplines?

Context and Modeling

GUIDING PRINCIPLE

The high school mathematics program must include situations, applications, and contemporary problems, often interdisciplinary in nature, that illustrate the usefulness of mathematics and employ mathematical modeling. Relevant contexts include worthwhile mathematical tasks, interesting applications, real-world opportunities to employ mathematical modeling, and problem-based lessons that motivate learning.

Traditional practice in high school mathematics is to spend days learning procedures by which to manipulate symbols before ever getting a glimpse at where and why these procedures have value. In the old days, it could have been argued that without facility with the procedures, there was no way to solve the problems and applications that used the procedures. In today's world, it is relating problem situations or applications to appropriate concepts that matters. It is mathematizing situations into equations or diagrams that matters. And it is the available technology, not procedural competence, that empowers students. For example, simply teaching statistical techniques in the abstract without tying these techniques to real-world situations and context, including, for example, predicting and analyzing election results, deprives our students of critical connections that strengthen both the depth of learning and the motivation to learn more.

Consider how much time is spent preparing students to solve the following exponential equation:

Solve for t: $16(.75)^t < 1$

Now consider students working in pairs, wrestling with the following problem:

> **You ingest 16 mg of a controlled substance at 8 a.m. Your body metabolizes 25% of the substance every hour. Will you pass a 4 p.m. drug test that requires a level of less than 1 mg? At what time could you first pass the test?**

In lieu of taking the log of both sides and remembering a bunch of properties for simplification, in today's world all students can use a free online scientific calculator to see what happens to the value of $16(.75)^t$ as time increases from, say, 1 to 12 hours and arrive at a very reasonable approximation for t. Or students can just as easily use a free online graphing calculator to home in on the intersection of $y = 16(.75)^t$ and $y = 1$ (see Figure 3.5). With the calculations relegated to technology, the important work *that technology cannot replace* becomes solving realistic problems by creating a mathematical model for the situation, understanding that a rate of metabolism of 25% means that 75% of the substance remains, creating a model—the equation—that mathematizes that understanding, and using your model to solve the problem and potentially extend it to solving other problems.

Ask students why they often connect positively with high school statistics and they will tell you that it is the real data, realistic contexts, and practical problems that allow them to apply their statistical understanding. Then consider how rarely we find realistic contexts and practical problems in a year of precalculus, unless one considers utterly contrived "applications" that are usually little more than exercises disguised with words. We provide many more contextual and modeling examples in Part 3 when we address pedagogy, assessment, technology, and modeling.

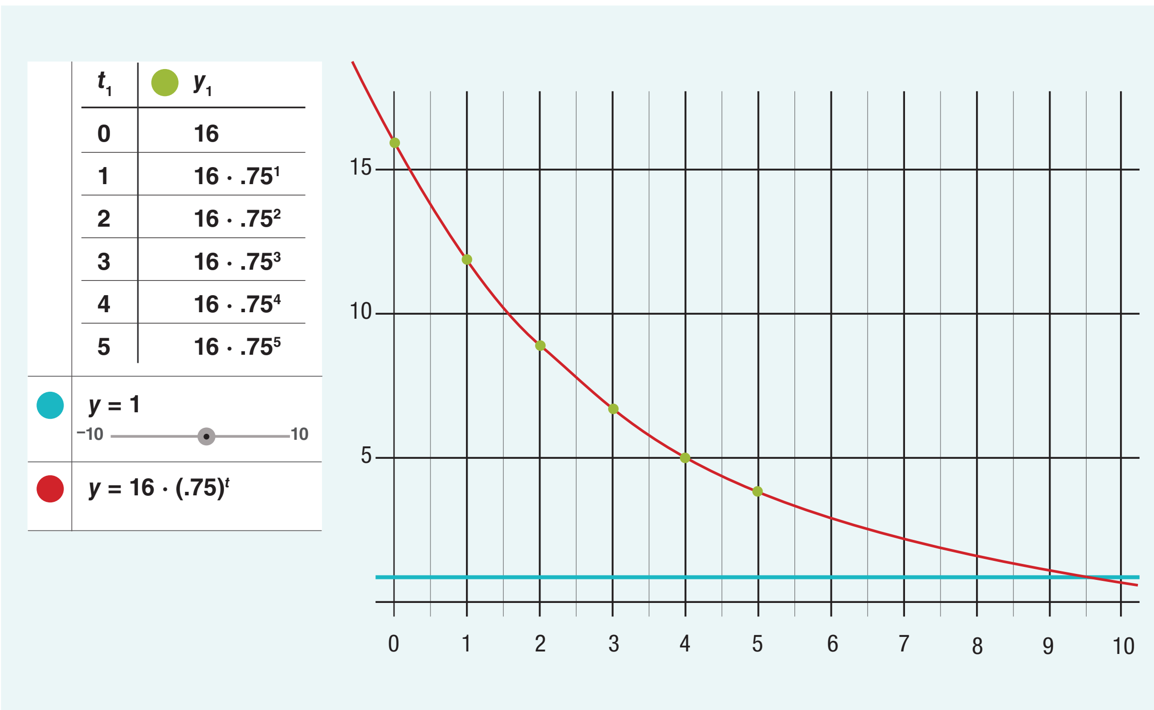

FIGURE 3.5

Questions to Ask Yourselves

- In recent lessons you have taught, how well have you balanced practical applications, mathematical modeling, and realistic contexts with the more typical skills of the lesson or unit?
- How often do you begin a lesson or a unit with a contextual example or application rather than focusing on skills first?
- Why, with all the access our students have to technological tools, do you think we still tend to emphasize skills over applications in our individual lessons, units, and assessments?

Assessment

GUIDING PRINCIPLE

The high school mathematics program must recognize assessment as an integral part of instruction. At its core, what we assess and how we assess it communicate most clearly what we value. Accordingly, assessments must include a balanced portfolio of strategically aligned, common, and high-quality summative unit assessments and an array of quizzes and benchmark tasks and other formative assessment techniques.

Another truism in life is that "what is inspected is respected." In schools, it is our assessments that tell students what we are inspecting and how we are inspecting it. These assessments clarify exactly what we deem important. And these assessments clarify what students need to respect by exerting the effort it takes to learn and master new content. How often do we hear, "Will this be on the test?" And good luck to any teacher who tries to maintain student interest after answering, "No, this is just enrichment."

But most importantly, we seek to debunk the idea of "first we teach, then we test, and finally we grade," as if these are three separate and distinct activities. Instead, we believe that instruction and assessment are not separate components of teaching, but integrally linked pieces of effective teaching and learning. That is, our decisions about what to teach and what to assess should go hand in hand. The alignment of our instructional goals, our instructional practices and tasks, and our assessments must be strong.

Effective teachers start the planning process by building clarity about the lesson goals and draft an exit ticket assessment that reflects the lesson goal *before* selecting tasks, examples, and exercises. Similarly, effective teachers *begin* unit planning with the unit assessment fully in mind and then adjust the unit assessment to reflect what they have learned from daily exit tickets.

We noted in the equity and access domain the practice of retesting. We look at this practice, as well as the critical role of common, high-quality unit assessments, in Chapter 7.

Questions to Ask Yourselves

- Is there a system of common, high-quality unit assessments for every course offered by your department? If not, why?
- Is it considered nonnegotiable that nearly every lesson concludes with an exit ticket or some other form of formative assessment to answer the simple question "What evidence do you have that this lesson was successful?" If not, why?
- Is there a system of retesting available to students for every unit assessment? Why or why not?

Technology

GUIDING PRINCIPLE

The high school mathematics program must make full use of technologies that increase the productivity of instruction and enrich students' experiences. The use of smartphones, calculators, computers, data-gathering tools and probes, interactive software, and real-time student data should be pervasive throughout instruction and assessment.

The availability of technology has changed the world. It has changed the workplace and both the availability of and expectations for jobs. It has opened new domains of study and supported incredible advances in medicine and science and commerce. It is technology—in conjunction with mathematics—that allows an Amazon order placed online at 10 a.m. to be delivered by 4 p.m. the same day or an Uber driver to arrive at your exact location in four minutes. It is technology, in the form of a single smartphone, that provides instantaneous, handheld access to an encyclopedia, a radio, a music collection, a mirror, a map, an atlas, a camera, magazines, a library of books, a calculator, a movie theater, a wireless phone, and so much more.

And yet there remains a wide range of practices and policies when it comes to technology in high school mathematics classrooms. We still fiercely debate whether students should be allowed access to their smartphones in mathematics classes, instead requiring them to lug around $129 graphing calculator bricks that have not decreased in price in nearly twenty years. We know that free interactive software such as Desmos activities is widely available, but it is not used everywhere. We know that high-stakes tests are increasingly delivered online and that available assessment technology can accurately score a range of constructed response items, but we cling to pencil-and-paper assessments that have scarcely changed in forty years.

We look at technology and the many ways it has become an indispensable component of effective teaching and learning of mathematics in Chapter 8.

Questions to Ask Yourselves

- What are your department policies on the use of technology to support teaching and learning in mathematics courses?
- What restrictions or limitations are in place for access to technology, and can they really be justified?
- How are video, online lessons, and free online supplemental resources used to support the teaching and learning of mathematics in every course in your department?

Adequate Time

GUIDING PRINCIPLE

The high school mathematics program must have adequate time to effectively meet the learning standards and implement the vision of teaching and learning that guides our work. It is not possible to accomplish what high school mathematics is expected to accomplish with anything less than 60 minutes of allocated time each day or the equivalent of 160 hours of classroom contact time for each course.

There is so much we know about how much time is needed, how much is typically available, and how it is typically spent. Most discussions about time start and end with "We don't control the length of a classroom period, so we have to live with what we are given." Our argument is that, from the perspective of both students' and society's needs, mathematics—when done right—is second in importance only to English/language arts. Treating mathematics, as is so often the case, as just another 46-minute period in a crowded day fails to recognize the scope of essential content and the critical need for interactive and collaborative practices that develop understanding, maintain interest, and support deeper learning.

When we conduct demonstration classes in which we are presented with a 45-minute class period, we start out stressed, and no matter how narrow our learning goal, we start looking at the clock. We eliminate opening activities, or we skimp on a discussion of the learning goal, or we cut short the time students need to solve and process, or we fall behind the clock and skip the all-important formative assessment. We cheat our students when we use time as the excuse for racing through the curriculum by showing, telling, and assigning practice.

When we have 60 minutes, we find ourselves much more likely to accomplish our goals. We have time to launch the lesson with a number talk or a cumulative review. We spend 2 minutes being clear about the day's learning goal and create an anticipatory set for the lesson. We even find time to joke or check in with students to support a safe environment that is conducive to learning. We then have time for two solid 20-minute lesson chunks before asking students to "turn and tell your partner what you learned today" and, finally, to complete a formative assessment exit ticket. Try doing all that in 45 minutes and no wonder teachers burn out!

We note the need for 160 hours for each course on the assumption that assemblies, field trips, mandated testing, delayed openings, and other assorted interruptions steal something close to 20 days in each course. Then, assuming that something close to 10 percent of instructional time is devoted to tests and quizzes, we end up with about 140 hours of non-assessment instructional time as the standard for which to strive.

When teachers have 45 minutes per day or 90-minute blocks every other day for 160 actual days without interruption, we are talking about 75 percent of the time that students with 60-minute periods are provided. There is no getting around the inequity among schools and districts, and thus for students, when the difference between 45 minutes and 60 minutes each day is so huge.

But of course, any discussion of time must also address how the time is spent. We urge teachers to launch 60-minute lessons with warm-up activities like number talks and cumulative review. We strongly urge all high school teachers to draw from released SAT and ACT examinations for warm-up tasks. We hope that going over homework is never a time-wasting 15- to 20-minute ordeal, but instead entails teachers posting the answers to the homework exercises or problems on the whiteboard and providing students with 5 minutes to review their work in pairs or triads, with particular attention to the most troublesome problems. Correct work for any problems that are still causing trouble can be easily displayed with a document camera and discussed before homework is collected, only to be recorded as completed. When two or three 15- or 20-minute instructional chunks draw from such options as lecture, games, video, online activities, selective practice, or project work, teachers acknowledge the limited attention span of adolescents and mix things up to maintain interest and to differentiate among approaches.

Questions to Ask Yourselves

- How many minutes are allocated to mathematics each day or every two days?
- How does this allocation compare to an average of 60 minutes per day?
- What discussions need to be conducted and what arguments need to be made to narrow the gap between current time allocations and 60 minutes per day?
- How do teachers use the time they have, and what consistency is there among members of the department on how time is allocated to specific elements of a typical lesson?

Professional Growth and Collaboration

GUIDING PRINCIPLE

The high school mathematics program must be supported by intensive attention to ongoing professional growth and collaboration among the mathematics teachers. In today's world, particularly in light of the technological opportunities and new differentiated courses being proposed, it is impossible to believe that any one teacher of mathematics can do it all. Accordingly, within every high school mathematics department there must be structures and practices including course committees, allocated time for collaborative planning, video and video reviews, and an expectation of collegial classroom visits and follow-up debrief discussions.

As teachers, our professional isolation within a dominant culture of "I'm okay, you're okay, don't bother me and I won't bother you" is the single greatest obstacle to change and improvement. Given the challenges of making mathematics work for *all* students, and the changes necessary to enact this goal, it is no longer acceptable to allow professional isolation to exist; nor can we continue the practice of treating our classrooms as personal castles surrounded by shark-infested moats. Instead, the dominant mantra must be "We're all in this together and what any one of us does affects what the rest of us can and cannot do."

Ask yourself, how often have you sat in a colleague's classroom for an entire lesson? When have you placed an iPad on a tripod or Swivl in the back of your room and just let it record a lesson that you later reviewed in the quiet of your personal space or that you shared with colleagues? Could you and a colleague interdependently plan and revise each unit? These are practices that we see in highly effective high schools. These are the practices that we return to in Chapter 10, where we focus on an implementation game plan.

The message we seek to convey is that changes of the magnitude we advocate take time and collaboration. Just as our goal statements pay homage to the "development of lifelong learners," as teachers we must commit ourselves to the same lifelong learning to stay current and vital in a changing world.

Accordingly, we end this section of guiding principles or domains of invigoration with one of our favorite quotes from NCTM's *Principles to Actions* (2014):

> In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for personal and collective professional growth toward effective teaching and learning of mathematics. A professional does not accept the status quo, even when it is reasonably good, and continually seeks to learn and grow. (99)

Questions to Ask Yourselves

- What collaborative structures are in place within your department to support a culture of sharing and professional interaction?
- What collaborative structures, such as course committees, video reviews, collegial classroom visits, and dedicated planning time, are not currently in place? Why do you think this is, and what can you do to change this?
- Do you observe colleagues on a regular basis and conduct a debrief session following each observation?

Conclusion

We would be completely irresponsible to suggest that all fourteen of these changes should, or even can, be tackled at once. Every high school we know has made progress on some of these changes and has miles to go on others. Change of this magnitude cannot be mandated "by the end of the year" or even "within the next two years." What we *do* know is that some of these changes enable other ones. For example, without addressing adequate time, acting on "less is more," or focusing more on the first four practice standards, real progress is unlikely. Similarly, without attending to building a shared sense of goals and purpose as well as a written vision, shifts in courses, content, instruction, and assessment are unlikely to be coordinated and effective. And without significant shifts in the departmental culture around professional growth and collaboration, teachers end up being told what to do without the requisite understanding, support, and buy-in.

The remainder of this book describes in greater detail what many of these proposed changes look like in classrooms, departments, and schools. Our goal is to provide guidance and examples for making these changes, for stimulating professional discussions, and for supporting the gradual implementation and institutionalization of each of these needed changes.

Guiding Questions

1. Of these fourteen domains, which ones do you think your department comes closest to meeting? In what ways is this the case?
2. Of these fourteen domains, which ones do you think your department is farthest from meeting? Why do you think this is the case? What specific steps can you take to change this?
3. Of these fourteen domains, and given that it is impossible to simultaneously address them all, which two or three do you and your colleagues believe are good places to start?