Teaching
for Thinking

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Fostering Mathematical Teaching Practices Through Reasoning Routines

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For Magdalene Lampert

Because teaching is learnable

And in honor of our parents,

John and Grace,

Bill and Kathy.

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The Power of Reasoning Routines

> **The question is not whether all students can succeed in mathematics but whether the adults organizing mathematics learning opportunities can alter traditional beliefs and practices to promote success for all.**
>
> —National Council of Teachers of Mathematics, *Principles to Actions*

Instructional routines are powerful vehicles for developing mathematical thinking and teaching practices. Their predictable design and repeatable nature make routines habit inducing, and over time those habits coalesce into a deliberate practice. We created our reasoning routines to develop specific mathematical practices in all students. It turns out those same routines also bring about change in mathematics teaching practices. In this chapter we reintroduce our routines for reasoning and how they are designed to create a collaborative learning environment that places a premium on mathematical thinking and reasoning. We then talk about five essential strategies woven into the fabric of the routines that can be leveraged to make the three shifts in practice necessary for teaching all students to think and reason mathematically (i.e., shifting more focus on to the thinking, stepping out of the middle, and supporting productive struggle).

What Are Reasoning Routines?

Teachers use many different types of routines in the classroom to support student learning. From management routines like clock buddies and daily bell ringers that keep students following the group plan to discourse routines like think-pair-share and stop-and-jots that support class discussions, all these routines have the same thing in common: they leverage the predictability of their design and their repeated nature to build effective habits of learning. Our reasoning routines, which are designed to develop mathematical practices in all students, are no different. The predictability of the designs for interaction in our reasoning routines frees up brain space so that students can use more of their brain power thinking and reasoning mathematically. Over time and with repeated use students develop powerful mathematical habits of thinking.

We introduced four reasoning routines in *Routines for Reasoning*: Capturing Quantities, Connecting Representations, Recognizing Repetition, and Three Reads. Each routine targets a specific standard for mathematical practice, but they all share common design features that keep a laser focus on math thinking and provide access to a wide range of learners. A brief description of each reasoning routine and a visual that shows the repeatable flow of each routine is given in Figure 2–1. The icons within each visual highlight the designs for interaction—how students engage with each other, the content, and the teacher—that remain the same within each routine.

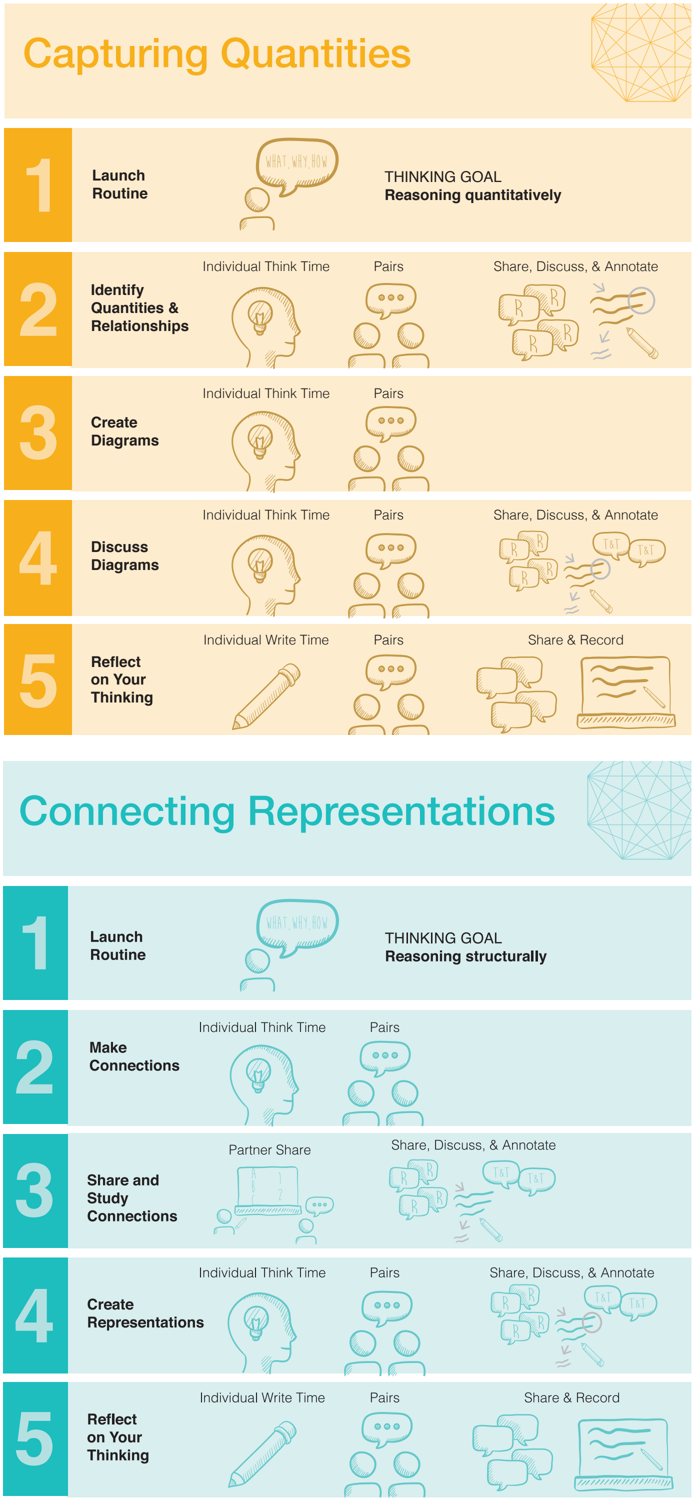
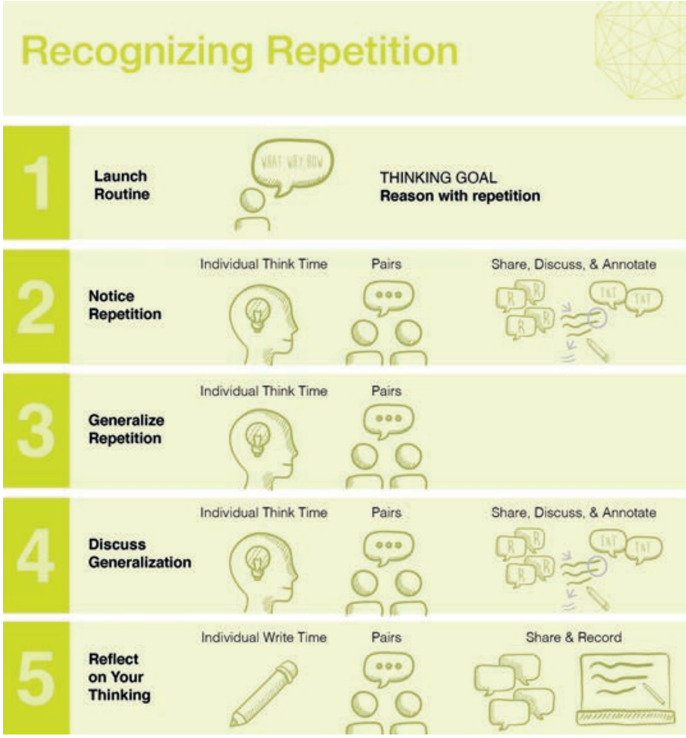


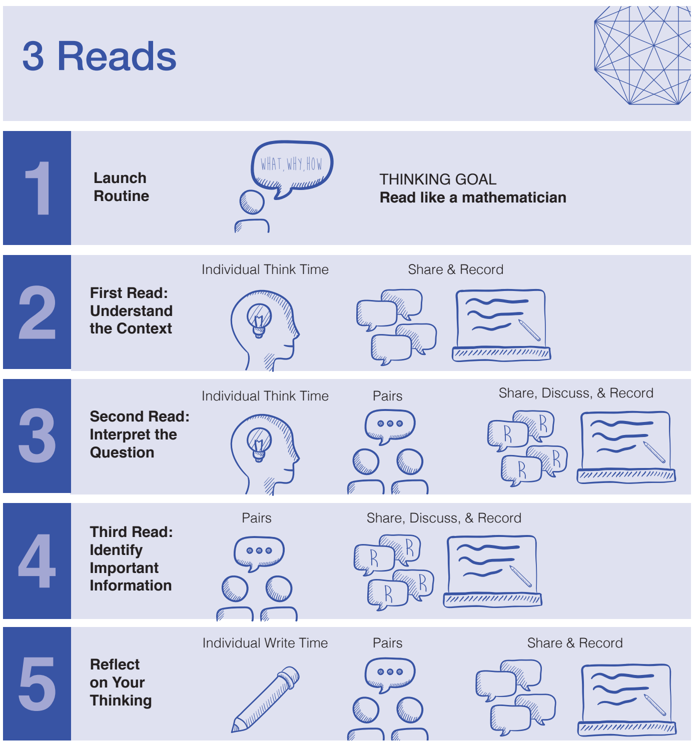
Figure 2–1

The goal of the Capturing Quantities reasoning routine is to develop quantitative reasoning (Common Core State Standards [CCSS] Standards for Mathematical Practice [SMP] 2, Reason abstractly and quantitively [National Governors Association Center for Best Practices, Council of Chief State School Officers 2010]). In the routine, students identify quantities and relationships in a problem situation, work with a partner to create a diagram that shows all the quantities and the relationships among them, then share diagrams in the full group and discuss where/how they see various quantities and relationships in the diagrams. Finally, students reflect on what they have learned about reasoning quantitatively.

The goal of the Connecting Representations reasoning routine is to develop structural thinking (CCSS SMP 7, Look for and make use of structure). In the routine, students analyze and then work with a partner to connect two different types of representations, share and study connections in the full group, then create, share, and discuss a missing representation. Finally, students reflect on what they have learned about thinking structurally. You saw a glimpse of this routine in Mr. Ryan's class in Chapter 1.



The goal of the Recognizing Repetition reasoning routine is to develop repeated reasoning (CCSS SMP 8, Look for and express regularity in repeated reasoning). In the routine, students engage in a counting, constructing, or calculating process and sense the regularity, share repetitions they are noticing, and then work with a partner to generalize the repetition. They then share and discuss generalizations in the full group, and finally, they reflect on what they have learned about reasoning through repetition.



The goal of the Three Reads reasoning routine is to make sense of math problems (CCSS SMP 1, Make sense of problems and persevere in solving them). In the routine, students read a word problem three times, each time for a different purpose. After the first read, students make sense of the context, answering, "What's the problem about?" After the second read, pairs restate the question in their own words, and share and discuss their rephrased questions. After the third read, students identify important information and as a group share and record information. Finally, students reflect on what they've learned about reading and interpreting a math problem.

Figure 2–1 (*continued*)

How Do Reasoning Routines Create a Math Thinking Learning Environment?

In Chapter 1 we said that thinking classrooms contain three key ingredients: they position students to think and reason mathematically, they provide time to think and reason, and they provide collaborative structures within which to think and reason. In this section we will explain how our reasoning routines create a math thinking learning environment.

Students Think and Reason Mathematically

Our reasoning routines are explicitly designed to develop mathematical thinking. Each routine is driven by a clearly articulated math thinking goal, makes use of ask-yourself questions and sentence frames and starters that prompt mathematical thinking, and ends with a meta-reflection so that students can name and solidify what they have learned about thinking like a mathematician.

Thinking Goal

Each routine for reasoning begins with the teacher clearly stating the type of thinking students will be developing as they engage in the routine. A thinking goal not only provides common language to talk about the mathematical thinking, it also communicates that the focus of the lesson is not the answer nor the steps taken, but rather sharing and explaining how students are making sense of the problem, what they think is mathematically important, the things they are paying attention to, and the questions they are asking themselves that are driving their problem solving. In short, the purpose of the lesson is to build a math thinking muscle. Some examples of thinking goals from the routines we just described include:

- Learn to identify quantities and relationships in problem situations (Capturing Quantities).
- Learn to connect numeric expressions to visuals (Connecting Representations).
- Learn to record and track repetition in your calculations (Recognizing Repetition).
- Learn to read a math problem three times to understand the context, to understand the question, and to identify important information (Three Reads).

Thinking Prompts

Two types of thinking prompts—ask-yourself questions and sentence starters and frames—are used throughout each reasoning routine to maintain focus on the thinking goal. Ask-yourself questions (e.g., "What are the important quantities in this situation?"

or "What would a mathematician think was important about this?") are posed to orient initial sense making of a math problem or representation. They shift the focus from "What should I do?" to "What would be helpful to consider?" Sentence frames and starters are used to ensure a focus on thinking when students begin talking with a partner or sharing in the full group. Rather than students starting with the answer they got and what they did, these prompts orient students to begin their conversations with what they are noticing or wondering (i.e., they begin with their thinking). We will talk more about both of these types of thinking prompts later in the chapter.

Meta-Reflection

Every reasoning routine ends with a meta-reflection in which students reflect on what they have learned about thinking like a mathematician. Sentence frames are provided to focus student reflections on the thinking (see Figure 2–2). After individual writing time, students discuss their reflections with a partner and the teacher listens in to select reflections to share and record in the full group. These recorded thinking takeaways become touchstones students can reference in future lessons.

- When looking for quantities in a word problem, I learned to _____.
- The next time I interpret a _____ I will look for _____.
- When looking for repetition, I learned to ask myself _____.

Figure 2–2

Students Have Time to Think and Reason Mathematically

Thinking takes time. Each reasoning routine centers on one high-demand task, giving students the time they need to make sense of and grapple with that problem, as well as the various ways their classmates are thinking about it. For example, in the Capturing Quantities routine, students identify the important quantities and relationships in a single word problem and represent them with a diagram. The teacher posts a range of diagrams and students analyze each one, identifying the quantities and relationships from the problem statement that are captured in the diagram as well as any implicit quantities or hidden relationships that can now be seen in the visual. In Recognizing Repetition, students engage in a counting, calculating, or constructing process and identify and generalize the regularity in their process, and then the class shares and discusses student generalizations and the underlying repetition.

Each routine is designed to build the habit of pausing to notice, interpret, and question before moving to problem solving. To push back against many students' instinct to grab some numbers and calculate, each reasoning routine shifts the focus away from answer-getting and toward making sense of a situation, problem, representation, and so on. For example, in Three Reads, students read a word problem three times, each time for a different purpose. The focus is on pausing and interpreting a math problem before jumping to solve it. Similarly, in Capturing Quantities, students first identify quantities and relationships in a problem situation before re-presenting them in a diagram.

Look at the infographics of the four reasoning routines in Figure 2–1 and you will see that the think-pair-share structure is used throughout the routines. Each time students are presented with a task, they get some individual think time to interpret the task before working with a partner. Additionally, students always have some private think time to make sense of a classmate's work before discussing with a partner and then in the full group. Any time a visual is introduced in a reasoning routine, be it student work under the document camera or a posted chart, students have individual think time to make sense of the image before working with a partner or answering a teacher question. Finally, whenever a teacher poses a question during discussions in the full group, there is always a pregnant pause for students to think before turning to a partner to discuss a response.

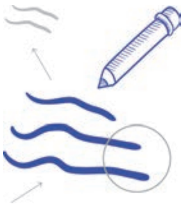
Students Think and Reason Mathematically Within Collaborative Structures

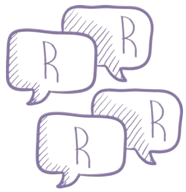
Although students have individual time to begin thinking throughout our reasoning routines, the bulk of the mathematical thinking and reasoning happens in student pairs. The purpose of individual think time in our routines is for students to ready themselves to work productively with a partner, not to come up with an answer that they check with their partner. The pair structure provides students a safe space to share initial noticings, ask and discuss questions, gain another perspective, and collaboratively work through mathematical ideas and language. Look at the infographics in Figure 2–1 and you will see that partner work is woven throughout each routine. Similarly, turn-and-talks during all full-group discussions in the routines provide processing space for ideas and language when students share and discuss each other's thinking. The teacher uses talk moves (more about those under "The Four Rs," on p. 33) to support collaborative sense making during full-group discussions as students are expected to listen to their classmates and explicitly prompted to revoice their ideas, adding more detail and increasingly precise language.

Now that you know what we mean by reasoning routines and how they are intentionally designed to provide time and collaborative structures in which to think and reason mathematically, we want to highlight five essential instructional strategies permeating our routines.

Five Essential Strategies

The five essential instructional strategies in our reasoning routines are:





I noticed...
so I knew...
so I looked for...
...connects to...
because...

- annotation
- ask-yourself questions
- Four Rs
- sentence starters and frames
- turn-and-talks.

We highlighted the first four essential strategies in *Routines for Reasoning* (Kelemanik, Lucenta, and Creighton 2016). Although we referenced turn-and-talks as a mainstay of language-rich classrooms and illustrated their use and impact in the vignettes of the routines, we did not explicitly label them as an essential strategy. We now include the turn-and-talk in our list of essential instructional strategies as we have seen time and again how a well-placed turn-and-talk during a full-group discussion can provide time and a safe, supportive structure in which students can work out their mathematical thinking and language. In addition, turn-and-talk is a very effective instructional strategy for stepping out of the middle and promoting collaborative sense making among students. In this way turn-and-talks are both a powerful student support and a strategy that teachers can leverage to develop a thinking classroom.

These five essential strategies promote the shifts in practice necessary for a thinking classroom—they maintain a focus on thinking, cause teachers to step out of the middle, and support productive struggle. Because these instructional strategies are baked into the design of our reasoning routines, the routines become effective vehicles for developing instructional practices critical for thinking classrooms.

In the next section, we will describe each essential strategy, how it keeps the focus on mathematical thinking, supports a range of learners, and helps teachers make the three critical shifts in practice necessary to teach mathematical thinking. We will also include instruction on how to get started with each strategy and highlight pitfalls to avoid.

Essential Strategy 1: Annotation

What Is Annotation?

Simply put, annotation is a visual representation of a student's verbalized mathematical thinking. During a full-group discussion, annotation can help students connect what they are hearing their classmates say to what they are seeing to make sense of a mathematical idea. For example, back in Chapter 1 Mr. Ryan used annotation to highlight, as a student was sharing, where that student had noticed "three groups of the same thing" in the visual and algebraic expression they had connected. Mr. Ryan's annotation included purposeful use of color (red to show the number of groups and blue to highlight what is in each group in both representations), words ("3 groups of") and symbols (he drew a rectangle above the x and two squares above the 2, and placed an x inside the rectangle and a 2 on top of the squares in each row.) (See Figure 2–3.)

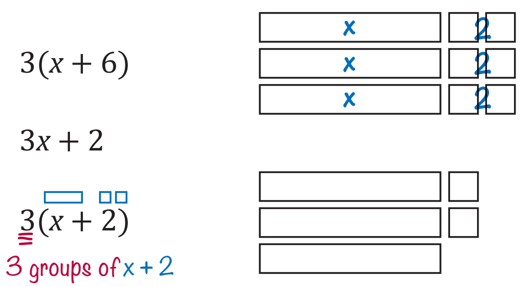


Figure 2–3

Annotation is not a one-time event; it unfolds as student thinking continues. In the case of Mr. Ryan's class, after the initial connection was made and annotated, he asked students to turn and talk about where they saw the parentheses in the first visual. As students shared and rephrased the idea that "the parentheses are like the rows," Mr. Ryan annotated the parentheses in red and then drew red parentheses around each row and added the numbers *1*, *2*, and *3* to the rows (see Figure 2–4).

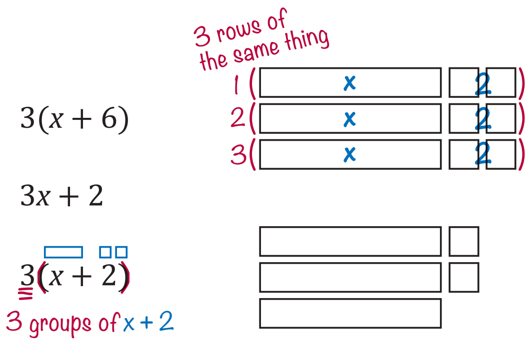


Figure 2–4

It is worth noting that we are talking here about annotation as an instructional move the teacher makes during full-group discussions to keep the focus on the mathematical thinking and to provide support for students who struggle to access and attend to auditory input. However, annotation is also a tool that mathematicians use as they work with pen in hand to make sense of something and/or communicate their thinking to others. Therefore, in some of our routines, such as Decide and Defend (see Chapter 3), students are explicitly prompted to annotate when they are writing or drawing or presenting their written work in the full group.

How Does Annotation Keep the Focus on the Mathematical Thinking?

Annotation is neither scribing verbatim what a student is saying nor simply writing down what *you* want to show students. In reasoning routines, the teacher uses annotation to highlight the particular avenue of thinking a student is using when sharing in the full group. This is clear in how Mr. Ryan visually connected the three "chunks" or "three groups of the same thing" with his annotation of the connected visual and algebraic expression. In this way he highlighted chunking, an important aspect of structural thinking.

How Does Annotation Support a Range of Math Learners?

Annotation provides an additional processing modality by connecting the verbal to the visual—whether that visual is written or gestured. The visual component is crucial for English learners and students who struggle to process information auditorily. Annotation also provides focus for students who struggle to orient visually, and it also creates residue for students who lose focus, become distracted, or simply walk into the middle of a class discussion. In all these ways annotation is a powerful support to help students focus on and make sense of their classmates' mathematical thinking.

How Does Annotation Help Teachers Shift Their Practice?

As we mentioned previously, the goal of teacher annotation is to highlight and support the mathematical thinking students share and discuss during full-group conversations. This might mean labeling quantities students identify in a context, or circling friendly "chunks" of an irregular shape, or writing down how a student "changes the form" of a number in their head when they are doing mental math, or using color-coding to track the impact of an estimate in a mathematical model. In this way, the teacher is calling attention to the mathematical thinking in student explanations.

When teachers annotate, they step out of the middle and privilege student thinking. The teacher remains silent as they annotate, making space for students to share and discuss their ideas with one another in the full group. Thus, this role of annotator sends the tacit message to students that their ideas matter and are being discussed.

It is all too common for students to check out during full-group conversations when they don't hear or don't yet understand what their classmates are saying. Once a student checks out, it is very difficult for them to check back into the conversation. Pointing to the part of a visual or place in a solution strategy that is being discussed can help students reorient to what is being said. When the teacher annotates the image about which students are talking, it not only connects the words being said to a visual but also provides residue of the conversation that a struggling student can consider as the class discussion unfolds. This is a powerful support that can keep students, especially those who are strong visual–spatial processors, struggling productively.

Getting Started with Teacher Annotation

In this section we share some advice on how to prepare for annotating, along with when and how to annotate. In Chapter 6 we will provide activities and advice that will help you increase your capacity to annotate your students' mathematical thinking in the moment.

Preparing to Annotate

Although annotation requires being responsive to student utterances in the moment, you can and should prepare for annotating. Just like an actor rehearses their lines before the theatre fills or a sports team practices plays before the big game, teachers should work through annotation options before they are standing at the board in front of the class trying to highlight mathematical thinking that a student is sharing in the moment.

Following these steps when you are planning your annotation will prepare you well for doing it in real time:

1. Articulate your thinking goal. Are you focusing on quantitative, structural, or repeated thinking? Is there a particular slice of that thinking you are targeting? For example, if students are finding a calculation shortcut (as we'll see in Contemplate Then Calculate in Chapter 4), do you want them focusing on structural thinking in general or is the goal to target changing the form of a numeric expression to make calculation easier?
2. Anticipate how students might work on the task. For example, if your structural thinking focus is "changing the form" and students are asked to find a calculation shortcut to determine the value of $2 + 25 + 3 + 35 + 4 + 45$, what are all the ways they might change the form of the expression or the individual terms to make it easier to calculate?
3. Practice annotating. Reproduce the expression multiple times and practice annotating and reannotating the various approaches you anticipated so that the annotation highlights how the form of the numbers and expression is being changed. Consider how you might use color, words, symbols, visuals, and so on. Figure 2–5 shows some examples.

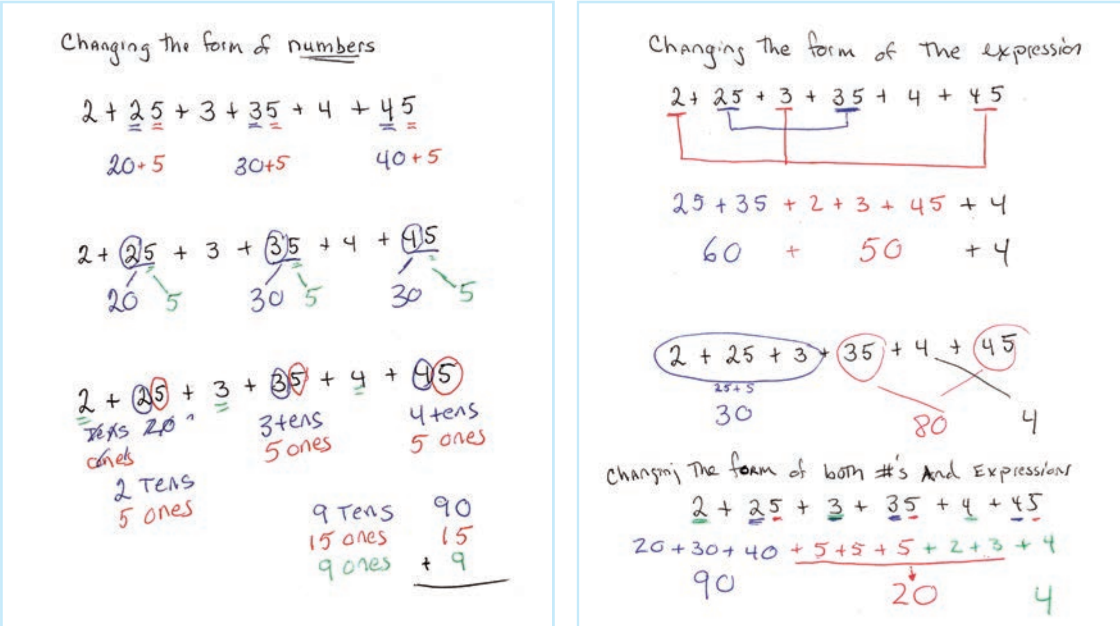


Figure 2–5

Planning for annotation is critical in that it will help you to maintain focus on your goal and better position you to hear the thinking in students' explanations and have some options for how to highlight that thinking. However, annotation is a responsive endeavor. It is crucial that in the moment you listen carefully to what your students are saying and annotate their thinking as it unfolds, instead of simply reproducing an annotation you prepared ahead of time. The goal of your annotation is to support your students as they make sense of and build on their classmates' mathematical thinking.

Implementing Annotating

In the classroom, the first step to annotating in the moment is listening carefully to what students are saying. That process begins before the class discussion as you tour the room listening in on partner conversations in advance of the full-group share. This allows you to get a sense of the thinking in the room and, ideally, select students to share particular lines of thought, as opposed to cold calling and not knowing what a student is going to say.

Annotation is most powerful when teachers allow an idea to percolate before annotating. So, during full-group discussions, we recommend that you do not make any annotations when a student idea is first shared. Feel free to point (or gesture) as the student initially shares their idea, then begin your annotation on the rephrase. We recommend waiting for a couple reasons. First, waiting to annotate on the rephrase provides every student in the class an opportunity to process what their classmate is saying. If you annotate as an idea is first being shared, you are in effect processing the idea, which robs the rest of the class of the opportunity to process what the students heard their classmate say. Pointing can help students track what is being talked about without processing it for them. Another reason to wait to annotate is that it gives you time to make sense of what the student said and think about how you can highlight the underlying mathematical thinking. When a student shares, listen for the mathematical thinking underlying their strategy. Listen for what it was they were noticing or knew that jump-started their approach. This prepares you to annotate more accurately and effectively during the rephrase. Do not feel as though you have to annotate everything on the first rephrase; you can (and should!) add more annotation as the discussion continues and the mathematical thinking unfolds.

Some practical suggestions follow:

- Use color, but not too many colors. Color is a powerful tool for highlighting connections. Using more than three colors will make the annotation too busy and cause visual noise. We try to stick with two easily distinguishable colors—red and blue.
- Use shading and symbols. If you do not have color (or even if you do!), lean on different types (dotted, solid, dashed) and thicknesses of lines as well as arrows, shapes, and grouping symbols, for example: { }, [ ], ( ).
- Include words, but not too many. Record any words or phrases that are critical to the discussion and thus would support students' participation in the discussion.
- Less is more. Start with a little annotation. You can always add more as the conversation continues.

Annotation Pitfalls

Following are three pitfalls that keep teacher annotation from being effective:

- Focusing on the answer. Annotation is meant to make mathematical thinking and reasoning visible. We make it a practice never to write down the answer, as we have found that as soon as we do, student attention shifts to what they got and if it is right, and the thinking and reasoning gets lost.
- Reproducing an annotation you practiced instead of annotating what a student is actually saying. Annotation helps students make sense of what their classmates are discussing, so if a student is saying one thing and you are annotating something different, it will only confuse students. And that can inadvertently send the message that the student ideas are inadequate or unimportant.

- Annotating after the student has stopped talking. If you find yourself continuing to annotate after the student has stopped speaking, in all likelihood you have moved beyond representing student thinking and are starting to show how you are thinking about the problem.

Essential Strategy 2: Ask-Yourself Questions

What Is an Ask-Yourself Question?

An ask-yourself question (AYQ) is a prompt designed to orient student thinking. AYQs are questions that productive math doers ask themselves to drive their mathematical sense making and problem solving. Initially the teacher asks these questions, but the goal is for students to internalize them and deploy them on their own while doing mathematics. Following is a list of sample AYQs:

- What can I count or measure in this situation?
- How can I chunk this expression into pieces I can connect with a visual?
- How can I generalize the repetition in my counting?
- What do I notice that might be mathematically important?
- What assumptions does this model make?
- Does the answer make sense? Does the process make sense? Are the calculations correct?

How Do Ask-Yourself Questions Keep the Focus on the Mathematical Thinking?

AYQs are designed to target mathematical thinking and reasoning. They are meant to be asked when students are making sense of contexts; analyzing representations, models, and arguments; working to gain traction on a seemingly intractable problem or when they have hit a dead end in their problem solving and need to switch gears. Simply put, they are meant to jump-start and restart math thinking and reasoning.

How Do Ask-Yourself Questions Support a Range of Math Learners?

AYQs provide students a way to start thinking when they read a problem and get overwhelmed by all the words, or look at a complicated expression and shut down, or glaze over when presented with a new mathematical representation. For example, students who have internalized AYQs can jump-start their problem solving by reading a problem once and asking themselves, *What's this problem about?*; reading the problem a second time and asking, *What's the question I am trying to answer?*; then reading the problem a third time and asking, *What are the important quantities and relationships in this situation?* Rather than shutting down when being presented with a complicated expression, students can ask themselves, *Can I change the form of this to make it easier to work with?* When making sense of a new mathematical representation, the student might ask, *What might be mathematically important about this?* These are questions that start students reasoning mathematically, without telling them how to think or what to do.

How Do Ask-Yourself Questions Help Teachers Shift Their Practice?

AYQs are subtle yet powerful drivers of mathematical thinking and reasoning in the classroom. As we mentioned previously, they are designed to prompt a particular avenue of math thinking, not telegraph the application of a concept or suggest a procedure to use. In this way, they help shift student attention to focus on mathematical thinking.

When teachers pose an AYQ, they step out of the middle and position students to think and reason on their own and with each other. In those instances when student pairs or the entire class gets stuck and looks to the teacher for help, the teacher can pose an AYQ, pause to give students a bit of individual think time, and then prompt them to turn and talk through a response to the question, once again stepping out of the middle and setting up collaborative sense making.

AYQs support productive struggle because they combat learned helplessness. When a teacher poses an AYQ, they provide students with a constructive thinking prompt, but more importantly, by not telling students what to do, they send the critical message that the student is capable of thinking and reasoning mathematically. Over time, AYQs serve as an internal compass for students by starting their mathematical thinking and jump-starting it if they are stuck. They empower students to drive their own mathematical thinking rather than depend on an external source (often well-intentioned teachers) for step-by-step directions and to persevere through nonroutine problems.

Getting Started with Ask-Yourself Questions

Preparing to Use Ask-Yourself Questions

The first step in preparing to use an AYQ is to get clear on your math thinking goal or the kind of mathematical thinking students might bring to the task you are posing. Choose or craft an AYQ that will prompt that type of thinking. For example, if your sixth-grade students are working on a ratios unit and struggling to distinguish between part-part and part-whole relationships in word problems, then posing an AYQ like "Ask yourself, '*What are the important quantities in this situation?*'" can help reorient them when they get stuck.

If you are working within one of our reasoning routines, AYQs will already be built into that routine. To prepare, look over the routine and make note of the AYQ(s) and consider what responses to the question(s) might sound like.

If you are crafting your own AYQs, keep in mind that an effective AYQ has two qualities: it orients to and develops mathematical thinking and it is neither too vague (e.g., "What do I know?") nor problem-specific ("What do I know about 17 and 5?"). These two qualities make AYQs usable across a wide range of concepts and problem types. When crafting AYQs for a specific avenue of thinking or a reasoning routine we are developing, we often ask ourselves, *What would a mathematician ask?*, and check to make sure the question doesn't just apply to this particular problem but can be used in a broad range of problem-solving situations.

Implementing Ask-Yourself Questions

Specific AYQs are built into all our reasoning routines. They typically tee up individual think time when students begin exploring a math task or when students pause to interpret a representation or a classmate's work. However, AYQs are also terrific prompts for getting students unstuck or when you want to orient students to a particular avenue of thinking.

If you are posing an AYQ to jump-start student thinking, pose the question and provide individual think time before transitioning students to partner work, a turn-and-talk, or full-group discussion. If you are posing an AYQ to restart thinking when a student pair or small group is stuck, ask the question and then walk away to give students time and space to think.

When posing an AYQ, you are not only trying to *prompt student thinking*, you are *teaching an aspect of mathematical thinking* (i.e., the kinds of questions mathematicians ask themselves when grappling with a math problem). Therefore, you must be explicit. When implementing an AYQ:

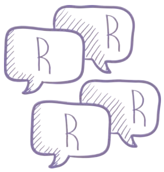
1. Project or write in a visible place the AYQ you pose.
2. Model the language in the AYQ.
3. Provide some individual think time for students to consider their response.
4. Reference an AYQ when students are stuck.
5. Periodically have students reflect on questions they are now learning to ask themselves to start and jump-start their mathematical thinking.
6. Keep a list of AYQs where students can see it and refer to it.

Ask-Yourself Questions Pitfalls

Pitfalls to consider when implementing AYQs might include the following:

- posing an AYQ as a teacher question and not explicitly as a question that students should learn to ask themselves independently
- not giving students time and space to consider their response to the AYQ before transitioning them to partner, group, or full-class sharing

- closing an open-ended AYQ by adding follow-up questions that shift the focus to an answer or procedure rather than the student thinking.



Essential Strategy 3: The Four Rs: repeat, rephrase, reword, and record

What Are the Four Rs?

The Four Rs are discourse moves that engage students in each other's mathematical thinking—but only if it is the students who are doing the talking and thinking. The Four Rs are:

1. Repeat. Students repeat a classmate's thinking exactly how it was stated.
2. Rephrase. Students rephrase a classmate's thinking in their own words, perhaps adding more detail.
3. Reword. Students reword a line of thinking using more precise mathematical language.
4. Record. The teacher records words or phrases being used by the class during full-group discussions.

Taken together, the Four Rs help students process and develop mathematical ideas and language and refine both so that they can articulate their thinking with increased precision.

How Do the Four Rs Keep the Focus on the Math Thinking?

In a thinking classroom the focus of the Four Rs is always on mathematical thinking. Students repeat, rephrase, and reword their classmates' thinking, not their answers or calculations. In this way, we help surface, process, develop, and refine how a classmate has come to approach a problem the way they did—what they attended to or knew or asked themselves.

How Do the Four Rs Support a Range of Math Learners?

Have you ever talked to yourself while thinking through a complex task? We have! Like us, many learners process ideas out loud. The Four Rs create space for students to process a classmate's thinking by rephrasing their idea or rewording it with more precision. When classmates repeat, rephrase, and reword each other's thinking, English learners and students who struggle to process auditory information get multiple passes at hearing that idea. The Four Rs also provide English learners regular opportunities to speak in math class. Recording language that students can reference during class discussions increases students' participation and helps increase English learners' language production, as does asking English learners to repeat what they heard a classmate say or reword a mathematical idea using language the teacher just modeled.

How Do the Four Rs Help Teachers Shift Practice?

The target of the Four Rs is always mathematical thinking and reasoning. Revoicing and having the opportunity to hear a classmate's line of reasoning multiple times provides all students the opportunity to articulate, process, develop, and refine mathematical thinking. The support the Four Rs provides students to work with and work through their classmates' mathematical thinking cannot be overstated.

The Four Rs is also a powerful tool for a teacher who is working to step out of the middle. When a teacher shifts from being the one who is repeating, rephrasing, and rewording to having students be the revoicers, they quite literally step out of the middle and explicitly orient students to each other's ideas. Repeating what our students say can be a hard habit to break. Many of us have been doing it for years, and for good reason. We repeat student ideas because we want to make sure everyone heard them and because we want to make sure they are stated as clearly and completely as possible. But when we are the ones revoicing student ideas, we inadvertently send the message to our students that they don't have to listen to each other, and this flies in the face of a classroom in which students are co-constructors of their mathematical ideas.

The Four Rs provide students who are struggling to follow a class discussion multiple opportunities to process what they are hearing. Having a struggling student rephrase a classmate's mathematical idea keeps them engaged and thinking productively about that idea. This repetition is critical for students who are working to make sense of a complicated mathematical idea.

Getting Started with the Four Rs

Preparing to Use the Four Rs

Although many of the decisions teachers make regarding the Four Rs are made in the moment, it's important to consider beforehand how to use them to further the thinking goal. This is because when you hear a student idea that is related to the goal, you will want to pause and use one or more of the Four Rs to provide students the support and space to process and refine that math idea and language. Taking time before the lesson to anticipate how students might talk about and around the goal will better prepare you to connect student utterances to the goal in the moment.

Implementing the Four Rs

In our reasoning routines, the Four Rs are used during full-class discussions. We also see these discourse moves used by students during partner and small-group work once they become a habit in the classroom. When you are thinking about when to use which *R*, consider the following.

After a student shares their mathematical thinking:

- If you think not everyone heard the idea, then ask that student or another student to repeat the line of thinking.
- If you are confident everyone heard the idea and needs to process it, then ask one or two students to rephrase the line of thinking in their own words.
- If you are confident that everyone heard and understands the idea but think that students could (and should!) communicate it more precisely using specific mathematical language, then ask students to reword the line of thinking. This may require you to prompt, provide, and/or model particular mathematical words and phrases.
- If there is specific language being used to communicate the idea that would be helpful for students to reference, then record those words or phrases publicly.

The first step to implementing the Four Rs is to not talk, and then think about why you were going to say what you were going to say. Knowing why you were going to speak will help you know which *R* to prompt.

Following are steps to integrating the Four Rs in the mathematical discourse in your classroom:

1. Preview the Four Rs. Before asking a student to share their thinking, for example, tell students that after a student shares, you will ask another student to rephrase their classmate's thinking.
2. Provide a sentence frame orienting to the thinking (e.g., "They noticed ______ so they ______" or "When they saw ______ it made them think ______ because ______") for students to use when they rephrase their classmate's idea.
3. Remind students to use the sentence frame.
4. Monitor the rephrase and ensure it is in fact a revoicing of their classmate's statement and not, for example, an explanation of what their own strategy was.

Be patient with students (and yourself!). It is not unusual the first couple times you ask for a rephrase to have the student explain their own thinking instead. Acknowledge it and gently ask for a rephrase of the original classmate's idea before moving on to this new idea. If you think that your students may bristle and ask why they have to rephrase or why you are no longer repeating or rewording important ideas, take the time to explain and ask them to help you step out of the middle and privilege their mathematical thinking and reasoning.

Four Rs Pitfalls

Pitfalls to consider when implementing the Four Rs might include the following:

- allowing the discussion to shift if students begin to rephrase but then talk about their own thinking, rather than reorienting to the original idea

- falling into the trap of repeating or rephrasing student thinking, instead of prompting students to be the ones doing the revoicing
- focusing the Four Rs on procedures and steps taken instead of the underlying mathematical thinking.

I noticed...
so I knew...
so I looked for...
...connects to...
because...

Essential Strategy 4: Sentence Starters and Frames

What Are Sentence Starters and Frames?

A sentence starter or frame is a skeleton of a sentence that helps students organize and communicate their thinking. A sentence starter like "A question I learned to ask myself when interpreting a diagram is _____" orients student communication. A sentence frame like "When I saw _____ it made me think _____ so I _____" helps students organize their thinking to communicate it more clearly. Both starters and frames give students a running start in communicating their idea.

How Do Sentence Starters and Frames Keep the Focus on the Math Thinking?

All the sentence starters and frames in our reasoning routines prompt mathematical thinking. Rather than focusing on answers and steps (e.g., "I got _____ by _____"), they focus on the mathematical thinking that led to the steps and answers (e.g., "We noticed _____ so we _____"). Whether the object of the starter or frame is the student themselves ("I saw _____ so I connected _____") or a classmate ("They noticed _____ so they _____"), the goal is always to uncover and communicate mathematical thinking and reasoning.

Sentence frames and starters can also be used to help students develop thinking language. For example, if you are working on quantitative reasoning with students, sentence starters like "The number of _____" and "The amount of _____" can help them shift from talking about the numbers in a math problem (e.g., 24 and 12) to naming the quantities those numbers represent (e.g., "the number of cookies with chips and the number of cookies without chips"). Likewise, if you are developing structural thinking, using a sentence frame like "I changed the form of _____ to _____" provides language for students to talk about this aspect of structural thinking. Following are some more sample sentence starters and frames we find helpful:

- They represented _____ by _____.
- Every time they _____.
- I agree/disagree with your defense because _____.
- We noticed _____ so we _____.
- An assumption this model makes is _____.

How Do Sentence Starters and Frames Support a Range of Math Learners?

For an English learner, starters and frames support language production and often provide the nudge they need to share their thinking in class. A sentence frame can help a student who struggles with executive functioning organize their thinking when they share orally in class. For students who struggle with anxiety, using a sentence starter or frame when turning to talk with a partner can mean the difference between speaking and not speaking. Although they are critical supports for some students, sentence starters and frames can help *all* students communicate their math thinking and reasoning.

How Do Sentence Starters and Frames Help Teachers Shift Practice?

All too often, when we ask students to share, they give their answer and say what they did, and omit how it was they came to do what they did (i.e., their mathematical thinking). Even when we explicitly ask students to "share their thinking," they almost always just tell us what they did. Sentence frames and starters push back against this instinct and force students to start at the beginning and share their initial sense making.

When we provide sentence starters and frames like "They noticed ______ so they ______" and "They represented ______ by ______," we step out of the middle and orient students to each other's thinking. These sentence frames send a clear message to students that it is their responsibility to be listening to their classmates' ideas and working to understand how they are thinking about the task at hand.

Sentence starters and frames support productive struggle by helping students to attend to, organize, and articulate their and their classmates' mathematical thinking and reasoning. When students share their approaches using sentence frames that prompt them to begin with what it was they noticed and how that led to their approach, the curtain is pulled back and what often feels like a mystery or a math trick to students begins to make sense (e.g., "Oh, you saw the multiplication and that made you think of area!"). When students are asked to rephrase a classmate's idea using a sentence frame that focuses on the thinking, they immediately begin to see why their classmate took that approach, and this will help them use that approach on a similar problem in the future.

Getting Started with Sentence Starters and Frames

Preparing to Use Sentence Starters and Frames

We use sentence frames and starters throughout all our reasoning routines whenever students are speaking to help them articulate their mathematical thinking and reasoning. If you are using a reasoning routine, look over these prompts and take note of the starters and frames used when pairs start talking, when students share in the full group, when students rephrase a classmate's idea, and when students reflect at the end of the routine.

If you are building sentence starters and frames into a lesson that doesn't use a reasoning routine, think about the type of thinking students might be using, then craft a sentence frame or starter that would help students articulate their thinking (i.e., what they noticed or wondered that drove their thinking). You will find that many of the starters and frames in our reasoning routines can be used in other lessons. We have found that they are the routine feature that students most readily transfer to other lessons. For example, after experiencing Contemplate Then Calculate (see Chapter 4) a couple of times, students naturally start with "I noticed ______" when sharing their thinking in other lessons.

Implementing Sentence Starters and Frames

We recommend using sentence starters and frames during these occasions:

- Students are sharing an idea.
- You want to orient students to a specific avenue of thinking.
- Students might need a starting point for discussion.
- Students are writing about their thinking.

It is critical that students don't see sentence starters and frames as optional or as a support for a subset of students. Although they are a critical support for some students, these prompts are intended to be used by all so that classroom discourse—be it in pairs, small groups, or the full class—focuses on mathematical thinking. Here are some steps to implementing sentence starters and frames in your lessons:

1. Project, record, or provide in writing the sentence frame or starter.
2. Model the sentence frame or starter.
3. Set the expectation that *all* students will use the frame or starter, because you want and expect all students to develop their mathematical thinking.
4. If students are sharing in the full group, remind them of the sentence frame or starter.
5. Hold students accountable for using the sentence frame or starter.

Sentence Starters and Frames Pitfalls

Pitfalls when using sentence frames and starters might include the following:

- using sentence frames and starters that do not prompt thinking
- providing sentence frames and starters for *some* students to use, not the entire class
- not holding students accountable for using them
- not listening carefully to how students use the frames and starters (for example, a student who says, "I noticed that I needed to multiply the two numbers and I got seventy-two" is not really saying what they noticed. They are beginning

with the sentence starter, but just saying what they did. In cases like this, ask the student to go back and say what it was that they noticed that caused them to multiply).

Essential Strategy 5: Turn-and-Talks

What Is a Turn-and-Talk?

Turn-and-talks are an opportunity for students to work out mathematical ideas and language with a partner. They differ from partner work in both scope and purpose. A turn-and-talk is relatively short (thirty seconds to three minutes) and is used to provide space to think out loud and gain an additional perspective. Partner work, on the other hand, is generally much longer (10–30 minutes) and is a structure for joint problem solving. Think of a turn-and-talk as a relatively short collaborative processing structure that provides every student the opportunity to speak and develop language and thinking. Because this processing is done out loud, turn-and-talks also allow the teacher to hear the range of thinking in the room, which is critical for selecting and sequencing student ideas.

How Do Turn-and-Talks Keep the Focus on the Math Thinking?

Turn-and-talks are used in reasoning routines during full-group discussions to provide students time and space to make sense of or drill down deeper into their classmate's thinking. These are not turn-and-tells, as in "Turn and tell your partner what you got for number 7." They are turn-and-talk about a particular line of reasoning (e.g., "Do you agree with their reasoning? Why or why not?" "What do you think they had to notice to use this strategy?" "Do you think this model is precise enough? Why or why not?"). We are fond of saying that if you give students a turn-and-talk, you have to give them something to talk about . . . a little mystery to figure out.

How Do Turn-and-Talks Support a Range of Math Learners?

Like the other essential strategies, turn-and-talks support a wide range of learning needs. For English learners, a turn-and-talk provides a safe structure in which to work out math ideas and work on the language to communicate those ideas, thereby increasing language production. For students whose strong suit is verbal processing, a turn-and-talk provides a space to talk through a mathematical idea. A turn-and-talk that is posed before sharing in the full group provides a hesitant speaker a safe place to work out an idea and practice articulating it. Building turn-and-talks into full-class discussions gives students who struggle to maintain focus times to stop and process what they are hearing.

How Do Turn-and-Talks Help Teachers Shift Practice?

Turn-and-talks can help teachers shift class discussions to focus on thinking, provided the prompt they have students talk about targets a mathematical line of reasoning or requires students to think.

A turn-and-talk literally forces the teacher to step out of the middle of a whole-class discussion and orients students to talk with a partner. During full-class discussions in a traditional classroom, the teacher asks one student a question, that student answers, then the teacher asks another question and another student answers, then the teacher asks a third question and a third (hopefully!) student answers, and the "discussion" continues along these lines. A turn-and-talk can be used to break this teacher-student-teacher-student-teacher-student dynamic that places the teacher at the center of the discussion. Instead of the teacher posing a question to the class and calling on a student to answer, the teacher poses the question and prompts a turn-and-talk so that every student in the room has an opportunity to consider the question and talk through a response with a partner.

Punctuating full-group discussions with turn-and-talks not only takes the teacher out of the middle and orients students to each other's ideas, but also provides critical processing time and space for students grappling to make sense of the task at hand. Regularly building turn-and-talks into full-class discussions can keep students who might tune out or give up engaged in the thinking.

Getting Started with Turn-and-Talks

Preparing to Use Turn-and-Talks

Preparing to use turn-and-talks is straightforward. Think about the questions you might pose when the class is sharing and discussing their ideas. Identify questions that will require students to think and reason versus telling an answer or simply recalling information. Consider crafting a sentence frame for students to use when they turn to talk with the partner.

Implementing Turn-and-Talks

Turn-and-Talks fall under two categories: those that teachers plan for ahead of time and those that they craft in the moment. Start by implementing planned turn-and-talks, and soon you will find yourself reaching for a turn-and-talk in the moment.

To get started, identify when in the lesson students will benefit from turning and talking. Then, craft a prompt to get them talking. When in the classroom, follow these steps:

1. Pose (and possibly record or project) a clear question or prompt.
2. Provide a sentence frame or starter to prompt partner talk.
3. Provide a time estimate (that you may adjust as you listen to students).

4. Listen to students as they discuss, and select and sequence responses.
5. Reconvene the class and remind students of the prompt.
6. Purposefully call on students to share their thinking and transition back to a full-group discussion.

Once you get comfortable with planned turn-and-talks, pay attention as student thinking develops and ask yourself when a turn-and-talk might be helpful to provide collaborative oral processing support. Here are four times we reach for a turn-and-talk:

1. When students need time and space to process a math idea and/or language before engaging in a full-group conversation, a turn-and-talk gives them that time and space.
2. When the teacher asks the class a question and no one raises their hand, a turn-and-talk provides the needed processing time.
3. When the teacher asks the class a question and all the students raise their hand, a turn-and-talk allows every student to answer.
4. When the teacher needs to hear how students are thinking to make an instructional decision, a turn-and-talk makes student thinking audible.

Turn-and-Talk Pitfalls

Pitfalls to using turn-and-talks effectively might include the following:

- Posing a vague turn-and-talk (e.g., "Turn and talk to your partner about what you heard"). Like a prompt that asks for information recall, a vague prompt is ineffective in prompting thinking and reasoning.
- Letting a turn-and-talk drag on too long. Typically, the room erupts when a turn-and-talk commences, and after a while the level of conversation begins to die down. This is the time to bring the group back together.
- Conferring with students during a turn-and-talk. The teacher's role during a turn-and-talk is to tour the room quickly, listening in on as many pair conversations as possible, so as to get a sense of the range of thinking in the room.

Chapter Summary

Although we laid out each of the five essential strategies individually, they are intended to be used together as a cohesive suite of supports in a thinking classroom. They work in tandem to keep the focus on the thinking, orient students to each other's ideas, and provide access and support so that all students are thinking and reasoning mathematically. Making these strategies a regular feature of your teaching practice will take time and attention, and the routines in this book can be great vehicles for this work. No matter which routine you begin with, you will find yourself posing AYQs to orient student thinking, annotating as students share their thinking, prompting students to repeat and rephrase their classmates' ideas, offering sentence starters and frames to focus and support communication, and using turn-and-talks to give students processing time and space. Over time, using these strategies becomes second nature in the routine, and you will find yourself migrating them into other lessons. We will talk more about mindfully building these practices and integrating them throughout all your teaching in the last chapter of the book.

The next two chapters introduce new reasoning routines. As you read about each routine, notice where the essential strategies are in the design. When reading the vignettes that show the routines in action, attend to how the teacher uses the strategies to get and keep all students working collaboratively to think and reason mathematically.

REFLECT ON YOUR READING

1. Reread the quote at the beginning of this chapter. What is one belief or practice you would like to alter to create a math thinking classroom?
2. How do reasoning routines promote mathematical thinking in all students?
3. Choose the essential strategy you would like to incorporate first in your teaching. Why did you choose that strategy? How might you implement it?