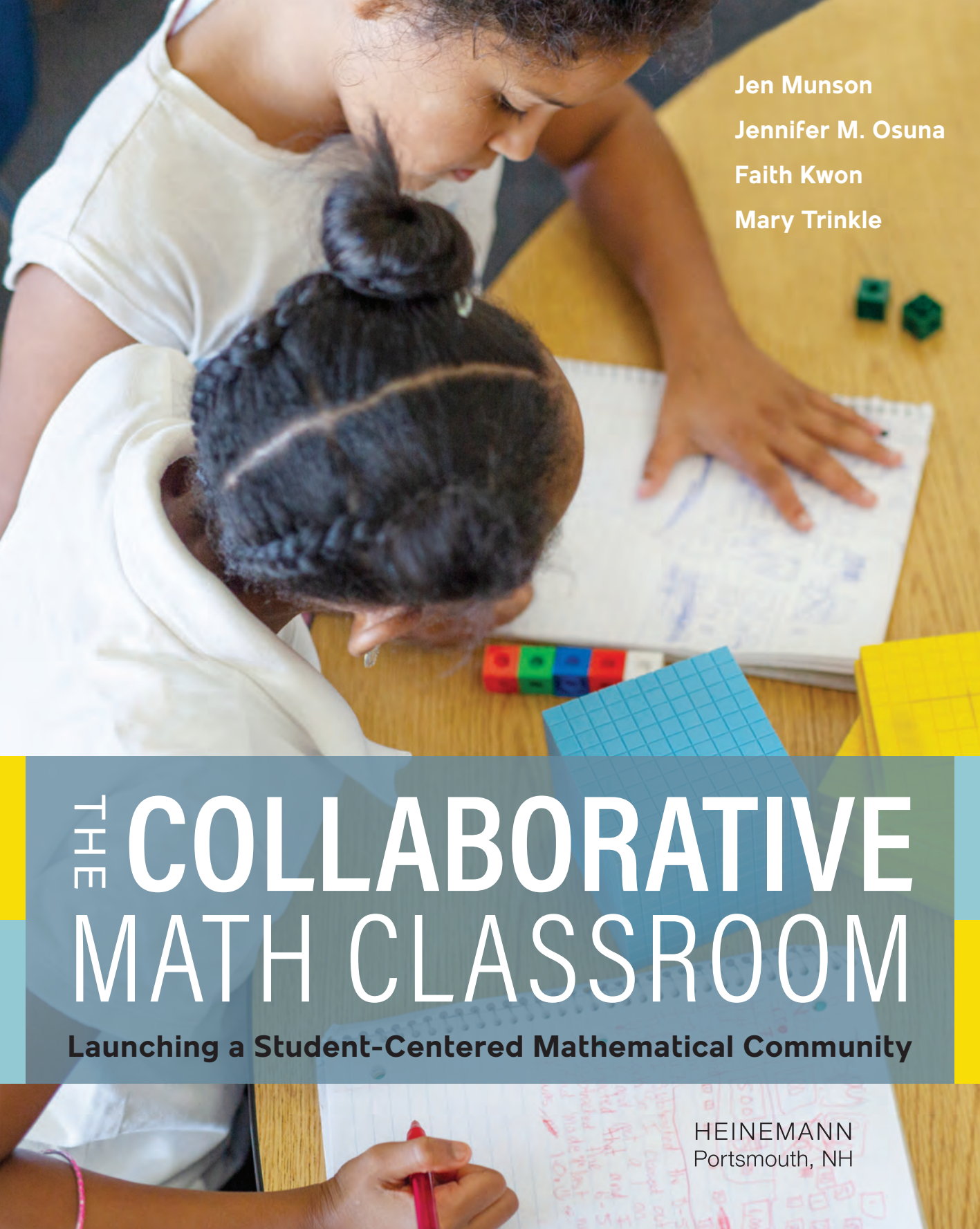

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THE COLLABORATIVE MATH CLASSROOM

Launching a Student-Centered Mathematical Community

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Introduction

Joel and Filemu sat on the carpet in one corner of the classroom surrounded by shelves of books and tools trying to decide what to do next. Their second-grade class had been asked to figure out how many T-shirts were left in their class's imagined T-shirt factory inventory. Joel and Filemu were in charge of the medium T-shirts. On the previous day they had 56 medium T-shirts in inventory, and today they were told that 37 of these T-shirts had been sold. How many medium T-shirts remained? Joel said to Filemu, "That's a lot. I don't know," and seemed to be stumped as they both stared at the problem. Filemu then erupted with an idea, "I know! Let's use cubes!" Joel's face softened, and with a "Yeah!," he hopped up to get a basket of snap cubes from the shelf behind them.

Joel suggested they make sticks of 10 cubes each, something they had done before, but Filemu was suddenly concerned. "But we only need 5 sticks of 10. It won't be fair," she said, as she noticed that one of them would get to make 2 sticks of 10 while the other would make 3. This was a serious problem for Filemu, but Joel pointed out that someone had to make the 6 extras to show fifty-*six*, not just fifty, T-shirts. Satisfied, they made their model, and then decided to act out selling the T-shirts. Joel would buy them from Filemu, and they would see how many they had left. Filemu immediately handed over 30 T-shirts, in the form of 3 sticks of 10 cubes, and then the pair faced the conundrum of how best to act out selling 7 more shirts from their remaining 26 T-shirts. They stared at the sticks of 10 and the 6 loose cubes. "What could we do?" Filemu asked Joel.

These two second graders had been working together for ten minutes and had already navigated several hurdles. They decided how to get started on solving the problem before them by agreeing on a tool to model the situation. They decided how to use that tool in a way that reflected and leveraged their understanding of base ten. They figured out how each of them could contribute equally to creating the model. And they decided to act out the action of the story to model selling T-shirts. When they confronted yet one more conceptual dilemma, Filemu turned not to their teacher, but to her partner to generate an idea of what to do next. This is complex work for a pair of seven-year-olds, but it is precisely the kind of work that leads to deep mathematical learning.

Learning Mathematics Is Fundamentally Social

The study of human learning and development is clear: learning is a social endeavor. We have long known that we learn by exploring and reflecting on our world with one another (Dewey 1985 [1916]). People learn through interaction with our environment, with one another, and, in this case, with the discipline of mathematics (National Academies of Sciences, Engineering, and Medicine 2018; National Research Council 1999). And yet, peek into the typical classroom and you are likely to see individual seats in clean rows, with students often working on assignments on their own with little talk allowed or encouraged. What a wasted opportunity to engage the curiosity of so many humans in the room! Why?

To put it plainly, it is harder to foster collaboration and inquiry and make sense of students' complex mathematical thinking than it is to put students in rows and ask for simple right and wrong answers. Although the former offers the possibility of greater engagement, more robust learning, deeper connection, and even joy, the latter is alluring because it is an environment that is easier to control. But it goes against what we know about how humans learn and, in practice, acts as a factory model that produces many students who dislike, avoid, and do not understand math. The results from the 2017 National Assessment of Educational Progress (NAEP) showed that only 34 percent of eighth-grade students performed at or above the Proficient level on the mathematics assessment. Fully half of Americans experience math anxiety (Boaler 2012); ask around and you are likely to hear fully competent adults claim that they are no good in math and often quite phobic of it.

In contrast, collaborative and cooperative approaches to learning mathematics that center reasoning and reflection are effective, engaging, and equitable for students, though ambitious work for teachers. It is worth putting in the hard work of creating a collaborative math classroom, full of mathematical inquiry and reflection, and in this book we will walk you through the steps of how to create one with and for your students.

Shifting Our Role

Engaging in mathematical practices as a classroom and making sense of ideas together requires intellectual risk-taking, creativity, and lots of talk. The National Council of Teachers of Mathematics has been calling for just such a

vision since 1989, with the Common Core State Standards jumping on board this train in 2010. And yet, moving toward creating math classrooms like this has been slow and difficult work. Shifting instructional practice means shifting our sense of what it means to teach mathematics well. For example, traditionally, the primary role of the teacher is to explain mathematics (Amit and Fried 2005). In the collaborative classroom, the teacher still explains mathematics, but that role becomes a secondary one, used judiciously in service of a new primary role—to cultivate mathematical inquiry and orchestrate mathematical discussions among students themselves (Herbel-Eisenmann, Wagner, and Cortes 2010; Stein et al. 2008).

When teachers shift roles toward supporting and orchestrating collaborative mathematics experiences among students, they also hand over more of the thinking work to students. Students take on the authority to author and evaluate mathematical ideas. During small-group or partner work, students take on even more of the thinking work since they are largely on their own while the teacher moves from group to group. Students also claim greater agency in how they participate in classroom tasks: they think about mathematics problems in different ways, and all of their ideas might be at play as students make sense of them together.

This can feel messy and complicated. Many interactions happen simultaneously and in a multitude of ways every minute. Control is no longer at the center of classroom management; instead, teachers and students create a particular kind of mathematical community together. It can be dizzying; so much so that it may seem easier to just put kids in rows or ask each student to complete their own worksheet. But we know this is not how people learn effectively.

How We Can Collaborate

In this book, we make five assumptions about you, our reader, and our role as teachers, researchers, professional developers, and writers in supporting your professional learning:

1. You are a professional.
2. You have knowledge that we don't.
3. We want to partner with you on your goals for your classroom and students.

4. Our experiences teaching in, learning from, and researching collaborative classrooms could support you in your journey.
5. We don't have all the answers. This is messy, beautiful work.

This means that we know that our experiences are not a mirror of yours, and therefore you may find that *your* collaborative math classroom looks different from the examples we describe in this book. Your journey to that classroom may have bumps and wiggles that echo ours but are specific to your context. And your deep knowledge of your students and context, along with the resources you have available, and understanding your own comfort zone are going to be critical to crafting a collaborative classroom. In this book, we draw on our collective experiences and expertise, and we, too, are learning all the time—from one another and from students—what it means to learn math together in community. If you want to launch a collaborative math classroom, but you're not sure how to wade into all this messiness productively and equitably, we can help. If you're a coach or administrator trying to promote school-wide change, we hope this book can break down the challenges into more manageable stages that you and your teams can think through together. Wherever you are, we invite you to join us.

How This Book Is Structured

We begin with setting a vision of teaching and learning mathematics and the principles that drive a collaborative mathematics classroom (Chapter 1). Since every classroom and context is different, we have designed the heart of this book around a set of goals, rather than as a series of steps. We know that no fixed set of rules, no formula, would ever work in every context. Indeed, having created collaborative classrooms again and again, we know that each group of children can require something different from us. But we have found that even with all these differences, some core goals guide us each year, and those goals tend to unfold in phases. This book is structured around four such phases (see Chapters 2–5). We begin with the premise that you might be starting this work at the beginning of a school year, and so you'll see that planning for your launch is the first of these phases (see Chapter 2). However, you can reimagine your math classroom at any point in the year. So, if it is February, and you have decided to dive in, we welcome you. From there, we examine three phases of working with students to establish and grow your collaborative mathematical

community: establishing a collaborative structure (Chapter 3), building and sharing authority (Chapter 4), and becoming a student-led mathematics community (Chapter 5).

Just as establishing a collaborative math classroom rolls out in phases, your reading can as well. We encourage you to read the first three chapters of this book as you begin to map out your vision and first steps and not yet get bogged down in what comes after. When you have gotten your classroom launched, you'll want to read more. You're a professional and you'll know when you've read just enough to get started and not so much that you feel daunted. We close Chapters 3, 4, and 5 with signs to celebrate in your classroom—signs that we sometimes overlook—and situations and questions that often arise at that phase. We want to be there with you as you do this work, so try some things in your class and come back to us when we can support you.

The final chapters of this book can be of particular use as you think through issues that come up when you are working on developing a new kind of community. In Chapter 6, we discuss what it means to maintain and sustain a collaborative community over an entire year. In Chapter 7, we grapple with the challenges teachers sometimes face when trying to share authority with their students. And in Chapter 8, we explore how you can communicate with and cultivate support from stakeholders, including parents, colleagues, and school leaders.

Throughout the book, you'll find vignettes from our real classrooms and notes from mathematics education research that we think you will find illuminating and useful in your work. Each is intended to offer yet another way to understand what it means to work together with students to do and learn mathematics.

Meet the Authors

As educators our stance is that all students deserve to have access to quality instruction that centers them as whole humans through an ethic of care and recognition that their identities and experiences are important aspects of learning. We recognize that racism, sexism, and other oppressive ideologies impact how students experience care and recognition in math class, and we hope to offer support with how to empower all students as mathematical thinkers and learners. The four of us have been collaborating as professionals on

creating and learning from collaborative elementary mathematics classrooms for six years, each of us in a different role and all of us in constant, ongoing conversation. In this book, you'll hear different voices when we want to draw attention to the different roles we have played, whether as teachers, researchers, or coaches.

Jen Munson

I am currently an assistant professor of learning sciences at Northwestern University, where my research focuses on how interactions support learning in the elementary mathematics classroom. I am particularly interested in how teachers interact with students during collaborative problem-solving and how teachers and math coaches collaborate in the classroom to support teacher learning. Previously, I worked as an elementary and middle school classroom teacher and then as a math coach in Pre-K through eighth-grade classrooms, supporting teachers to foster equitable, productive math classrooms. Spending time in hundreds of classrooms over the years, I learned much from teachers and children about what was possible, and I still had many, many questions. These questions led me to my graduate work at Stanford University, where I worked with Jenny and we met Faith and Mary. Together, we have been learning from what happened in their classrooms as they did the messy and joyful work of sharing authority with their young students, learning we hope to share with you in this book.

Jennifer M. Osuna

I am an associate professor of education at Stanford University. My research focuses on student identity formation in collaborative mathematics classrooms. In particular, my research examines how students work together in whole-class discussions, small-group work, and partner work, examining how they negotiate participation and its effects on how they come to see themselves and others as learners and doers of mathematics. I serve as principal investigator to the research study that housed this writing team, and I led a research group that partnered with five elementary school teachers interested in learning to teach mathematics through collaborative learning activities. Faith and Mary were two of the teachers we partnered with at the time, and their powerful instructional practices were foundational to what our team learned about inclusive, productive, and thriving collaborative mathematics classrooms for young learners.

Faith Kwon

I am a doctoral student at Stanford University, studying race, inequality, and language in education, teacher education, and math education. My research interests include humanizing support for preservice and inservice teachers, particularly for teachers with marginalized identities and justice commitments. Prior to beginning my doctoral studies, I was a first- and second-grade teacher, and a district instructional coach and professional developer supporting early career teachers in developing student-centered, inquiry-based instructional practices. Mary is a longtime colleague and thought partner in the work of building classroom spaces of mathematical collaboration, an inquiry that was hugely supported by Jenny and Jen in my years as a classroom teacher and now graduate student and researcher.

Mary Trinkle

I serve as the elementary math coach for my school district, where I facilitate a program with cohorts of teachers who participate in professional development and coaching to implement collaborative math classrooms. I strive to create trusting and collaborative relationships with teachers and leaders to best support growing teacher practice in ways that center adult learning and the process it takes to be vulnerable in learning. Before this, I taught fourth grade and kindergarten, mostly the former. As a classroom teacher, my ultimate goal was to cultivate a learning environment that centered student inquiry and exploration. My work with Jenny, Jen, and Faith has been essential to my growth, first as a classroom teacher when we met and now as a coach. Our conversations and collaboration have been instrumental to my understanding and implementation of collaborative math classrooms and facilitating professional learning with teachers.

1

A Vision of Teaching and Learning Mathematics

Notes from the Classroom

Let's begin our journey by peeking into Faith and Mary's thinking about their former classrooms, a first- and fourth-grade classroom, respectively, where students learned to work on mathematics together in ways that centered collaboration, mutuality, and self-direction.

FAITH: "What Does It Look Like? What Does It Sound Like?"

Back from recess, my students gather in a circle on the rug, some of them stopping by their tables to hang their sweater or taking a quick drink at the sink on their way. There is a low buzz as they settle; I smile and acknowledge their excitement, then begin by collectively reviewing what we already know about "collaborating" ("working together"), because we will need to collaborate ("work together!") to play our new game. We ask ourselves, "What does it look like when we're collaborating?" and "What does it sound like?" After

some think time and one or two initial responses, I point to our anchor chart and other visuals as cues, knowing that eventually they will be able to cue each other and remind themselves how we collaborate and play games together. In a fishbowl, I model the game with a student, simultaneously prompting the students in the circle around us to revoice the steps. I ask if a partnership can model for all of us what it looks like to get started. As Jimena and Eva get up and begin the work of choosing a good spot to work, gathering tools and materials for the game, and negotiating who goes first, I am voicing over their actions and directing the students' attention to the moves they are making as collaborators and mathematicians.

The students indicate with a thumbs up that they are ready to begin this work as well, and I invite them to get started. As this is happening, I am also taking quick scans of the room, taking note of things I may want to bring up as we debrief our game time, but also glimmers of how I know this time will look and sound with time, intentionality, and practice. They may not all be collaborating productively yet, but with repetition, reminders, and a shared vision of how we work together, I know that's where we'll land.

MARY: Buzzing with Collaboration

Looking around my classroom in March, I saw students engaged in their math work the way that I had envisioned in August. Partners were spread across the room, some sitting at tables, leaning in and pointing to each other's work, and some lying on the carpet with manipulatives all around them and with a thinking expression as their partner explained how the materials related to the fraction problem about sandwiches. A few students stood around the board to reread the problem, pausing after each sentence to discuss the context. A student stared out the window and then looked at their work with an excited expression. I could feel in my body that amazing things were happening. I stood there for several moments to soak it in.

In the moment, I decided we would debrief how collaboration was going. I wanted my students to revel in their success. I called for a cleanup and we gathered around the carpet. I asked students, "How did your partnerships go today?" the same way that I had so many times this year. But today was different. One student shared about changing their strategy after seeing what their partner did; another student shared about getting two different answers and having to think through what to do next together; another student shared about a conflict that they had about where to sit and reflected on what might be a better way to solve their problem tomorrow. My students were not only

being productive in their partnerships, they were talking about the mathematical work they had done. They connected the work of collaboration to their mathematical thinking. We had reached the magical moment where everything clicked and the classroom was buzzing with collaboration!

A Vision of Teaching and Learning Mathematics

When we imagine a mathematics classroom, certain elements are likely to come to mind: a teacher at the front of the room presenting and explaining procedures, students raising their hands if they have a question or to share the right answer on individual assignments. In this book, we offer an alternative vision for a collaborative mathematics classroom. The role of the teacher is not so much to explain as it is to curate and facilitate rich mathematical experiences that students engage in with one another. Students' mathematical thinking and curiosities drive the day's work; teachers elicit, probe, and help to nudge students' thinking, wonder along with students, and help to create a classroom where students feel seen and heard as mathematical thinkers, learners, and community members. In this vision, the classroom is more than a place where students acquire important content knowledge about mathematics; instead, it is a place of both learning and becoming. Rather than simply learning mathematics, students learn to become young mathematicians together—posing and making sense of problems, directing their work together, and experiencing themselves and one another as mathematical contributors and thinkers.

A collaborative elementary mathematics classroom is an active, lively, and often joyful place marked by particular kinds of social interaction: children move about the room, finding and using the tools they need to make sense of

the ideas at play, whether counting cubes, rulers, base ten blocks, or just paper and markers. Students sit in twos, threes, and fours at tables or sprawled on the rug, discussing strategies for solving problems. You might find it hard to spot the teacher; she's kneeling next to a pair of students as they build ten-sticks with cubes and talking with them about their work, clarifying and nudging their thinking forward.

Realizing this vision of a collaborative classroom might seem overwhelmingly complex at first glance, but we can break this vision down in terms of specific kinds of important social interactions. Social interactions are the building blocks of human activity, including the doing and learning of math. How students interact with the classroom space (the *environment*), with *one another*, and with *mathematics* defines the classroom community and each person's part in it. Rather than a classroom made up of students working on individual assignments or listening to a teacher's explanation, a collaborative classroom involves a variety of peer interactions, including asking questions, explaining ideas, expressing and resolving disagreements, gaining attention, making decisions together, and many more. In other words, unlike a traditional mathematics classroom, a collaborative classroom involves *dialogue.* Opportunities for making sense of mathematics, as well as for identifying positively as a young mathematician, emerge through that dialogue, both in the whole class and small groups. In this sense, collaborative mathematics classrooms can be thought of in terms of particular kinds of social interactions that support student-directed mathematical explorations and dialogue. In this book, we break down collaborative classrooms into their component building blocks and offer detailed and actionable goals to support you in creating such a classroom for you and your students. First, we'll identify some underlying principles to guide our work.

Seven Principles of a Collaborative Mathematics Classroom

We've identified seven principles to guide a collaborative mathematics classroom. In this section we'll highlight each principle and some of the research it is based on.

1. Teachers trust students and themselves. Students trust themselves, each other, and the teacher.

A thriving and productive collaborative classroom is fundamentally built on trust. Teachers must trust that students are earnest in their efforts to engage

with the environment, each other, and mathematics. And teachers must also learn to trust themselves by leaning into their curiosity, getting to know their students, responding to students' ideas, and recognizing that not everything needs to go perfectly each day. Students must be able to trust that their teacher will both accept them as they are and guide them in learning how to participate in community with others, that they do have ideas that are worth bringing to their classroom community, and that their peers will respond to their ideas in ways that make them feel seen and heard.

Research Note **Understanding Off-Task Activity.** Students are earnest participants in classroom activity, even when it seems that they are off task. In a study of the functions of off-task activity on mathematical collaborations, researchers found that most instances of off-task activity actually promoted the collaboration. For example, off-task activity helped a student struggling to contribute to a collaborative work gain the attention of peers and join the collaboration; off-task activity also helped to recruit students into collaboration, to warm up to the work by connecting students, and to resist becoming overpowered by others (Langer-Osuna et al. 2020). Teachers who had the opportunity to attend to and notice the functions of such social negotiations during collaborative mathematics tasks shifted their interpretive lens for making sense of and responding to student behavior, noticing that students were earnest in their attempts to connect with one another and to get on the same page about the task at hand and that teachers could trust those efforts (Langer-Osuna and Munson, under review).

2. Students feel safe to bring their whole selves to doing and learning mathematics.

Trust begets both a sense of agency and a sense of safety. Students feel they can bring their whole selves, including their interests, ideas, curiosities, and experiences, to investigating mathematics with their peers. Sharing one's thinking and wonderings with others takes vulnerability, and, in a space of safety, such willingness to share and contribute fosters rich, productive discussions about the big ideas of mathematics.

Research Note **Autonomy and Identity in Math Classrooms.** A yearlong case study of student engagement in a collaborative mathematics classroom found that in a classroom where students had a great deal of autonomy in choosing their classroom participation, and where accountability practices were grounded in student-led negotiations, students were able to bring in valued identities to their mathematical work, increasing engagement and the formation of a positive mathematical identity (Langer-Osuna 2015).

3. All student voices are worthy.

A collaborative classroom needs diverse ideas to promote the authentic explorations, discussions, debates, and dilemmas that promote deep understanding. Even incorrect ideas can spark

Research Note **Collaboration Supports Inclusion.** Decades of research in complex instruction (Cohen and Lotan 2014; Featherstone et al. 2011) have shown the value of mathematical group work structured to support inclusive participation by all students. Cobb (1995) termed this structure as multivocal, and showed that, in multivocal small-group configurations, students engaged in more productive exchanges. Research by Mercer and colleagues (Mercer and Howe 2012; Mercer, Wegerif, and Dawes 1999) has similarly found that multivocality, central to classroom dialogue, helps students to reach greater learning gains.

aha moments when students wrestle with them while considering other ideas. Further, classrooms, when seen as spaces of becoming, require students to learn to use their voice, to explore and practice voicing their own ideas, and to know that their ideas are always worth considering.

4. Teachers center the needs and voices of vulnerable students.

To have authentic dialogue, students must learn to share intellectual authority to voice and discuss mathematical ideas jointly. Particular students might struggle to be seen and heard in the classroom for a variety of reasons, ranging from issues of status to cultural and social identities to challenges related to attention deficit hyperactivity disorder, autism spectrum disorder, or trauma. By centering the needs and voices of vulnerable students, the classroom creates the space for more inclusive dynamics that ultimately support richer mathematical dialogue.

Research Note **Centering the Voices of Marginalized Students.** In a yearlong case study of a collaborative mathematics classroom, Langer-Osuna (2011) found that students' interpretations of and responses to a peer's directives during collaborative work were shaped by gender, ultimately marginalizing the female group leader, while further centering the male group leader. Turner and colleagues (2013) examined a classroom whose participation structures for whole-class discussion explicitly centered marginalized students—in particular, English learners. By centering English language learners through norms such as inviting Spanish language, relying on the classroom community to serve as translators, and expecting monolingual English speakers to attend to and work to make sense of Spanish language contributions, the teacher helped to foster robust and inclusive dialogue.

5. Mathematics focuses on reasoning, sensemaking, and big mathematical ideas.

Students in collaborative math classrooms make sense of mathematics. By exploring and investigating big ideas, such as place value, addition and subtraction, fractions, or shapes, students have the opportunity to reason with one another and develop conceptual understanding. For example, students might add two numbers by first representing the problem by building ten-sticks out of linking cubes and then discussing what happens when the remaining ones of each addend are joined together, creating a new ten-stick. These students have the opportunity to not simply memorize addition facts but to explore what

Research Note **Focusing on Big Ideas.** Research by Jo Boaler has highlighted the many benefits of collaborative mathematics classrooms that focus on the big ideas of mathematics. In a study that compared a traditional mathematics program to a collaborative one, Boaler and Staples (2008) found that a focus on both collaboration and big ideas fostered students' understanding and enjoyment of mathematics. Further, this approach increased mathematics achievement. In more recent work, Boaler has found that a focus on big ideas supports mathematical exploration and dialogue (see the Mindset Mathematics series [Boaler, Munson, and Williams 2017–2022] and www.youcubed.org for resources).

place value is by constructing tens out of ones and reasoning about what it means to join two quantities through the mathematical act of addition. Students are able to *see* addition, investigate quantity, and make sense of place value.

6. Mathematical tasks invite and value multiple voices, conceptions, and strategies.

Because a collaborative mathematics classroom is based on exploration and dialogue, it matters that multiple ideas are at play. Simple, procedural questions meant to practice a single traditional algorithm provide little to discuss beyond how to do the procedure. Open tasks in a collaborative classroom enable discussion of multiple strategies. For example, even a simple addition expression like 32 + 67 can invite multiple ideas when students are encouraged to solve it in whatever way makes sense to them. One student might share that they imagined the problem as 30 + 60 and then 2 + 7, while another student might say that they imagined the problem as 32 + 70 and then removed the extra 3. Sharing why their strategies make sense and lead to the same answer provides rich opportunities for students to dig more deeply into numbers and also allows them to experience themselves, and others, as mathematical thinkers, learners, and community members.

> **Research Note** **Exploratory Tasks.** In research from the QUASAR project from the University of Pittsburgh (see *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development* by Stein et al. 2009), researchers found that tasks that were explorative and nonalgorithmic in nature, as well as tasks that required students to make connections between mathematical ideas, were most effective at supporting learning. See *Five Practices for Orchestrating Productive Mathematical Discussion* by Smith and Stein (2011), for support in how to implement tasks that support whole-class mathematical discussions.

7. The physical environment is designed to serve students and their work with one another and mathematics.

A collaborative mathematics classroom is active, hands-on, and student led. The physical environment must be conducive to collaboration, exploration, and even a bit of a break when needed. This means that students need table arrangements that facilitate conversation and pathways around the classroom that allow them to get up, choose needed materials, and find spaces in the room where they can investigate their ideas. A collaborative classroom can get somewhat noisy and students, especially those sensitive to stimulation, may also need a quiet corner to take a moment to think on their own before rejoining their group.

Sharing Authority in a Collaborative Mathematics Classroom

In the introduction, we highlighted the centrality of shared authority in collaborative classrooms. What does that mean? Certainly the teacher remains ultimately in the driver's seat for many important reasons, but, in classrooms that share authority, aspects of the thinking work, as well as making decisions about how to work as a team, become shared between the teacher and students and among students themselves. In this section, we clarify the forms of authority that teachers must continue to hold and the specific ways they share authority with students.

Teachers' Authority

Teachers still ultimately choose what topics students will focus on, what tasks they will engage in, and what products will be turned in and assessed, as well as set expectations for work and the culture of the classroom. But by sharing authority with students, teachers shift their roles in a few important ways. They shift from being "tellers" to being *facilitators* of productive thinking and *orchestrators* of discussions by eliciting and probing students' ideas. Teachers also shift from disciplinarians to community builders, creating classroom routines with and for students that support smooth and productive collaboration and supporting student-to-student communication through a variety of resources, such as language stems or strategies for conflict resolution.

Students' Authority

Research shows us that humans learn best by constructing ideas and understandings actively, through investigations, discussions, reflections, and other aspects of inquiry. We learn best with many opportunities to make decisions, follow our curiosity, and try out possible solution paths that make sense to us. Such forms of learning not only lead to deeper understandings, but are also empowering, leading students to becoming self-directed learners invested in understanding. Students need the agency to make sense of mathematics and to engage in tasks in ways that make sense to them, developing greater capacities and understandings and more sophisticated choices. Students also need intellectual authority: the opportunity to author ideas, evaluate the ideas of others, and determine whether particular ideas make sense and are reasonable.

Interactions Are the Building Blocks of Collaboration

Classroom dynamics that empower learners arise from particular kinds of interactions that foster productive, robust, and inclusive collaboration. Research on social interactions in mathematics classrooms highlights the importance of interactions that foster (1) mathematical investigations and explorations, (2) student-led dialogue and discussion about the big ideas in mathematics, (3) respectful disagreement among students and ways to resolve issues that arise with methods that are grounded in mathematical principles, and (4) mutual recognition and regard among students as mathematical thinkers, learners, and community members.

Three strands of interactions structure this book: interactions with the environment, interactions with others, and interactions with mathematics. We will bring in research to ground this vision, highlighting connections to studies on authority, collaboration, and interaction in elementary mathematics classrooms.

Interactions with the Environment

Interactions with the environment make up the physical elements of collaboration. These include how students are oriented toward one another through the arrangement of desks, tables, and open spaces. Interactions with the environment also include how students navigate in the classroom, choosing and reaching work spaces and available resources such as manipulatives and other tools. The goal is an environment that allows for easy navigation, organization, and collaboration. Even young children should be able to walk over to a resource table, easily choose among a range of resources, and put materials away in an organized fashion. Seating arrangements should facilitate sharing resources and attending to one another's talk. (For example, round tables support eye gaze, body orientation, and side-by-side interactions that promote fairly sharing and using resources while discussing ideas with one another.)

Both classroom environments featured in "Notes from the Classroom" supported collaboration in a variety of ways. In both classrooms, the carpet served as a gathering space for reflection and practice. Faith modeled important interactions in a fishbowl, while her first graders practiced with one another before going off to interact in such ways on their own. The carpet also served as a transition space between coming back inside from an outdoor

Faith's Classroom Environment

activity and moving into the mathematics block of the day. In Mary's classroom, space was used flexibly, as students chose where to work, creating workspaces out of tables or even the floor in the corner of the room. How students interacted within the classroom environment in part shaped what was possible for their mathematical work.

Interactions with Others

In a collaborative classroom, students regularly interact with one another. They explain their thinking, ask one another questions, and make a variety of joint decisions. They seek and offer attention, listen to one another, share resources, and resolve conflicts as they arise. In doing so, they not only make sense of mathematics together, but also develop important relationships and learn to act as a community. Students experience themselves and one another as legitimate thinkers, learners, and community members. Teachers can introduce, model, and practice with students particular kinds of interactions that support joint attention, shared meaning making, and mutual recognition.

In Faith's classroom, the teacher and students discussed the questions "What does it look like when we're collaborating?" and "What does it sound like?" Faith also offered and pointed to a classroom anchor chart and other visuals to support collaborative interactions. Faith also used a fishbowl approach to model interactions, inviting Jimena and Eva to model the work of choosing a good spot to work, gathering tools and materials for the game, and negotiating who goes first. As they did so, Faith named their actions aloud to support peers' noticing of the moves they were making as collaborators and mathematicians. Whether through modeling and practicing valued interactions in preparation for a particular activity or engaged in reflective discussions about their own partnerships, students learned to interact with intentionality and in ways that fostered productive, mutual engagement in deep mathematical thinking.

Interactions with Mathematics

Of course, at the heart of a collaborative mathematics classroom is interaction with mathematical ideas. Working with one another in a conducive environment, students confront, explore, and investigate big mathematical ideas, building conceptual understanding, investigating strategies for solving problems, and considering how to mathematize their world. These learning experiences lay a strong foundation for more advanced mathematics in the later years and allow students to develop procedural fluency with understanding. Students also have the opportunity to come to understand mathematics and develop a deeper appreciation for mathematics' elegance, beauty, and power.

As we saw in Mary's description, she noted that, in her classroom, at times students came up with and implemented possible solution paths together, and other times they each tried their own ideas and then shared their thinking to come to consensus, challenge one another, or deepen their own understanding. This is rich, conceptual work laden with learning opportunities that allow students both to *understand* mathematics and experience themselves and one another as mathematical thinkers, learners, and community members.

Progression of Goals: Building a Collaborative Classroom over Time

Moving toward this vision is ongoing work. In the next four chapters, we offer goals and strategies for this work in four phases: before the school year (or, more generally, before you launch this work in your classroom, which may begin later than the start of the school year), establishing a collaborative structure, building and sharing authority, and becoming a student-led mathematics community.

We invite you to join us on this exciting journey to create active, robust collaborative classroom communities. Let's get started!

Progression of Goals for Fostering a Collaborative Math Classroom

BEFORE THE SCHOOL YEAR (Phase 1)
Chapter 2 offers guidance on how to develop and address goals for planning before you begin to introduce collaborative structures into your classroom.

ESTABLISHING A COLLABORATIVE STRUCTURE (Phase 2)
Chapter 3 addresses goals for how to begin the work with your students and establish collaborative structures from the start.

BUILDING AND SHARING AUTHORITY (Phase 3)
Chapter 4 continues the work into the school year, focused on goals for building and sharing authority between teacher and students and among students.

BECOMING A STUDENT-LED MATHEMATICS COMMUNITY (Phase 4)
Chapter 5 builds on the work from Chapter 4, deepening and expanding classroom social interactions toward becoming a student-led mathematics community.